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THE CARTAN CONNECTION AS AN INVARIANT OF DIFFERENTIAL SYSTEM

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Abstract. We suggest the invariant geometric description of the system of ordinary arbitrary order differential equations achieved by the reduction of this system to the reductive Cartan connection in some naturally arising principal bundle.

1. Introduction

The principal aim of this article is the research of the invariant properties of the system of ordinary arbitrary order differential equations

$$\frac{d^p x^a}{(dx^1)^p} = S_p^a \left(x^1, x^b, \frac{dx^b}{dx^1}, \dots, \frac{d^{p-1}x^b}{(dx^1)^{p-1}} \right) \tag{1}$$

$a, b, c, d = \overline{2, n}; i, j, k, l = \overline{1, n}; p = 3, 4, \dots$, defined on the smooth arbitrary dimensional manifold V_n.

Such equations for small orders were studied by S.Lie, E.Cartan [1],A.Tresse [2]. The geometric theory of the equation on the plain was worked out by N.V.Stepanov [3]. The constructions created by L.E.Evtushik [4] also have direct applications to the geometry of differential equations. For example in [4], the reductive Cartan connection is joined to the ordinary third order differential system, which is of special interest to our research.

The general problem of constructing the geometric theory of the systems of ordinary arbitrary order differential equations determined on the arbitrary dimensional smooth manifold was first formulated by the author of this article in [5]-[8].

System (1) is treated as a cross-section

$$s : S^{p-1}(V_n) \longrightarrow S^p(V_n) \tag{2}$$

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of the bundle of tangent p-elements $S^p(V_n)$ over the space of (p-1)-elements of tangency $S^{p-1}(V_n)$ in the terms of the theory of fibre spaces.

The second order is excluded as it is beyond the general scheme of research. The fact that classical affine connection space is characterized by the second order system of geodesics holds out a hope of constructing some generalized analogy of the geometry of higher order differential systems in the terms of fibre spaces with connection.

The following structural equations

$$d\omega^k = \omega^i \wedge \omega_i^k;$$

$$d\omega_l^k = \omega_l^i \wedge \omega_i^k + \omega^i \wedge \omega_{li}^k; \tag{3}$$

$$d\omega_{l_1\ldots l_s}^k = \sum_{\alpha=1}^{s} \frac{s!}{\alpha!(s-\alpha)!}\omega_{(l_1\ldots l_\alpha}^i \wedge \omega_{l_{\alpha+1}\ldots l_s)i}^k + \omega^i \wedge \omega_{l_1\ldots l_s i}^k,\ s=\overline{2,p-1}$$

of the principal higher order frame bundle $H^p(V_n)$ are initial.

The equations

$$\omega_{(p)}^a = V_{pk}^{a0}\omega^k + V_{pb}^{a1}\omega_{(1)}^b + \cdots + V_{pb}^{ap-1}\omega_{(p-1)}^b, \tag{4}$$
$$\omega_{(p)}^a = \omega_{\underbrace{1\ldots1}_{p}}^a$$

are equal to system (1) or to cross-section (2) in the terms of the structural forms of the frame bundle $H^p(V_n)$.

The closed differential-geometric structure interpreted as the non-reductive Cartan connection in some principal bundle was constructed in [6]. The invariant geometric description of system (1) offered in this article is achieved by its reduction to the structure of the reductive Cartan connection in some sub-bundle of the frame bundle $H^P(V_n)$ with a very interesting structural group. Moreover the integral curves of this system are realized as geodesics of the associated connection. This fact allows us to consider the invariant description of arbitrary order differential systems in the terms of associated Cartan connection as a generalization of the description of the second order systems in the terms of affine connections.

The invariant construction of the reduction being considered is secured by the modern differential-algebraic method which comprises the exterior differentiation of equations (4) with application of algebraic Cartan lemmata and by the group-theoretic invariant method of prolongations and scopes in the combination of the reduction of the higher order frame bundle.

2. The reductive Cartan connection associated with the differential system

The above mentioned construction of the connection is based on the complex and multiple reduction of the higher order frames accompanying the stage by stage partial differential prolongation of structural equations (4). Thus the following Proposition 1 is the result of p-stage partial prolongation of equations (4) accompanied by fixing p-jet's components of cross-section (2) at each stage.

Proposition 1. *Ordinary differential higher order system (1) generates the correlations*

$$\omega^a_{b(1)} \equiv \omega^a_{b(2)} \equiv \omega^a_{b(p-1)} \equiv \omega^1_{111} \equiv \cdots \equiv \omega^1_{(p-1)} \equiv$$

$$\equiv \omega^1_c \equiv \omega^1_{c1} \equiv \omega^1_{c11} \equiv \omega^a_{bc} \equiv \omega^a_{(p)} \equiv 0$$

$$mod(\omega^1_{11}, \omega^1, \omega^d, \omega^d_{(1)}, \omega^d_{(p-1)}) \tag{5}$$

connecting the canonical p-order frame bundle differential forms.

Taking into account correlations (5) in equations (3) we receive:

$$d\omega^1 = \omega^1 \wedge \omega^1_1 + \Omega^1;$$

$$d\omega^1_1 = \omega^1 \wedge \omega^1_{11} + \Omega^1_1;$$

$$d\omega^1_{11} = \omega^1_1 \wedge \omega^1_{11} + \Omega^1_{11};$$

$$d\omega^a = \omega^b \wedge \omega^a_b + \omega^1 \wedge \omega^a_{(1)};$$

$$d\omega^a_b = \omega^c_b \wedge \omega^a_c + \frac{p-1}{2}\delta^a_b \omega^1 \wedge \omega^1_{11} + \Omega^a_b; \tag{6}$$

$$d\omega^a_{(1)} = \omega^b_{(1)} \wedge (\omega^a_b - \delta^a_b \omega^1_1) + \frac{p-1}{2}\omega^a \wedge \omega^1_{11} + \omega^1 \wedge \omega^a_{(2)} + \Omega^a_{(1)};$$

$$d\omega^a_{(s)} = \omega^b_{(s)} \wedge (\omega^a_b - s\delta^a_b \omega^1_1) + \frac{s(p-s)}{2}\omega^a_{(s-1)} \wedge \omega^1_{11} + \omega^1 \wedge \omega^a_{(s+1)} + \Omega^a_{(s)};\ s = \overline{2, p-2};$$

$$d\omega^a_{(p-1)} = \omega^b_{(p-1)} \wedge (\omega^a_b - (p-1)\delta^a_b \omega^1_1) + \frac{p-1}{2}\omega^a_{(p-2)} \wedge \omega^1_{11} + \Omega^a_{(p-1)},$$

where forms $\{\Omega\}$ are expressed only over forms $\omega^1_{11}, \omega^1, \omega^c, \omega^c_{(1)}, \ldots, \omega^c_{(p-1)}$.

It is obvious that in accordance with (6) we can obtain the structural equations $d\omega^a_b = \omega^c_b \wedge \omega^a_c$ of the invariant forms ω^a_b of the general linear group $GL(n-1, R)$ by fixing $\omega^1_1 = \omega^1_{11} = \omega^1 = \omega^c = \omega^c_{(1)} = \cdots = \omega^c_{(p-1)} = 0$.

Therefore structural equations (6) describe some principal bundle Q with the structural group $GL(n-1, R)$ over the manifold W with the basic forms

$$\omega^1_{11}, \omega^i, \omega^c_{(1)}, \ldots, \omega^c_{(p-1)}.$$

Then $\{\Omega\}$ become the semi-basic forms.

Let's construct the extension of Q by extending its structural group $GL(n-1, R)$ preliminarily noting that it is easy to verify the justice of the following statement:

Proposition 2. *Structural equations (6) are identified with equations*

$$d\Theta^1 = \Theta^1 \wedge \Theta^1_1;$$

$$d\Theta^1_1 = \Theta^1 \wedge \Theta^1_{11};$$

$$d\Theta^1_{11} = \Theta^1_1 \wedge \Theta^1_{11};$$

$$d\Theta^a = \Theta^1 \wedge \Theta^a_1 + \Theta^b \wedge \Theta^a_b;$$

$$d\Theta^a_b = \Theta^c_b \wedge \Theta^a_c + \frac{p-1}{2}\delta^a_b \Theta^1 \wedge \Theta^1_{11}; \tag{7}$$

$$d\Theta^a_{(1)} = \Theta^b_{(1)} \wedge (\Theta^a_b - \delta^a_b \Theta^1_1) + \frac{p-1}{2}\Theta^a \wedge \Theta^1_{11} + \Theta^1 \wedge \Theta^a_{(2)};$$

$$d\Theta^a_{(s)} = \Theta^b_{(s)} \wedge (\Theta^a_b - s\delta^a_b \Theta^1_1) + \frac{s(p-s)}{2}\Theta^a_{(s-1)} \wedge \Theta^1_{11} + \Theta^1 \wedge \Theta^a_{(s+1)};\ s = \overline{2, p-2};$$

$$d\Theta^a_{(p-1)} = \Theta^b_{(p-1)} \wedge (\Theta^a_b - (p-1)\delta^a_b \Theta^1_1) + \frac{p-1}{2}\Theta^a_{(p-2)} \wedge \Theta^1_{11}$$

of some $n + (p-1)(n-1) + (n-1)^2 + 2$ - dimensional Lie group G^{p-1}_n under the condition $\Omega \equiv 0$.

Extending the structural group $GL(n-1, R)$ of the principal bundle Q to the group G^{p-1}_n we obtain the extension $Q^{G^{p-1}_n}$ of Q. As it follows from (7) the homogeneous space $F = G^{p-1}_n / GL(n-1, R)$ is reductive. The dimension of F is equal to the dimension of W. According the Cartan-Laptev theorem, the system of Pfaff forms $\omega^1_1, \omega^1_{11}, \omega^a_b, \omega^1, \omega^c, \omega^c_{(1)}, \ldots, \omega^c_{(p-1)}$ connected by the structural equations (6) determines the fundamental group connection in the principal bundle $Q^{G^{p-1}_n}$ with curvature-torsion semi-basic forms $\{\Omega\}$. Thus, we have constructed the bundle $Q^{G^{p-1}_n} \supset Q$ with the connection associated with the ordinary arbitrary order differential system. At the same time there arises the situation typical of the Cartan connection [9],[10], when structural equations (6) may be interpreted as the structural equations of reductive Cartan connection forms in the principal bundle $Q^{G^{p-1}_n}$ limited on the sub-bundle Q, which proves the justice of the following:

Theorem 1. *The higher order ordinary differential system induces the reductive Cartan connection with limited on Q connection forms*

$$\omega^1_1, \omega^1_{11}, \omega^a_b, \omega^1, \omega^c, \omega^c_{(1)}, \ldots, \omega^c_{(p-1)}$$

and structural equations (6) in the principal bundle

$$Q^{G^{p-1}_n}(W, G^{p-1}_n) \supset Q(W, GL(n-1, R)).$$

Besides Ω^a_b are the curvature forms and the rest of forms $\{\Omega\}$ are the torsion forms of connection; ω^a_b are the forms of linear connection in the bundle Q with $\Omega^a_b + \frac{p-1}{2}\delta^a_b \omega^1 \wedge \omega^1_{11}$ curvature forms.

3. Integral curves as geodesics of Cartan connection

Introducing the relative coordinates of the first order element of tangency $s_1^a = \frac{v_1^a}{v_1^1}$, where v_1^i are the relative coordinates of the tangent vector in the current point of the curve $x = \lambda(t) \in V_n$, we receive for this curve: $\omega^a = V_1^a dt; \omega^1 = V_1^1 dt$ and therefore $\omega^a = s_1^a \omega^1$. The exterior differentiation of the last correlations results in $ds_1^a + s_1^b \omega_b^a - s_1^a \omega_1^1 - s_1^a s_1^b \omega_b^1 + \omega_{(1)}^a = s_2^a \omega^1$. Prolonging this process by taking into account the structural equations (3) of the higher order frame bundle and applicating the induction method we obtain the following:

$$\Delta s_0^a = s_1^a \omega^1; \Delta s_0^1 = s_1^1 \omega^1 = \omega^1 (s_1^1 = \frac{v_1^1}{v_1^1} = 1);$$

$$\Delta s_1^a = s_2^a \omega^1; \Delta s_2^a = s_3^a \omega^1; \ldots; \Delta s_p^a = s_{p+1}^a \omega^1,$$

where

$$\Delta s_0^1 = \omega^1; \Delta s_0^a = \omega^a;$$

$$\Delta s_1^a = ds_1^a + s_1^b (\omega_b^a - \delta_b^a \omega_1^1) - s_1^a s_1^b \omega_b^1 + \omega_{(1)}^a;$$

$$\Delta s_p^a = ds_p^a + \sum_{r=1}^{p} \frac{1}{r!} \sum_{q=0}^{p-r} (p-q)! \sum_{\alpha_1 + \cdots + \alpha_r = p-q} \frac{1}{\alpha_1 !} s_{\alpha_1}^{b_1} \cdot \cdots \cdot \frac{1}{\alpha_r !} s_{\alpha_r}^{b_r} \cdot$$
$$\cdot \left(\frac{p!}{q!(p-q)!} \omega_{b_1 \ldots b_r (q)}^a - \frac{p!}{(q+1)!(p-q-1)!} \delta_{b_r}^a \omega_{b_1 \ldots b_{r-1}(q+1)}^1 \right) -$$
$$-(p+1)! \sum_{r=1}^{p} \frac{1}{(r+1)!} \sum_{\alpha_1 + \cdots + \alpha_{r+1} = p+1} \frac{1}{\alpha_1 !} s_{\alpha_1}^{b_1} \cdot \cdots \cdot \frac{1}{\alpha_{r+1}!} s_{\alpha_{r+1}}^{b_{r+1}} \delta_{b_{r+1}}^a \omega_{b_1 \ldots b_r}^1 + \omega_{(p)}^a. \tag{8}$$

In the frames $\omega^1 = dx^1; \omega^a = dx^a; \omega_k^i = \cdots = \omega_{k_1, \ldots, k_p}^i = 0$ from (6) we have: $\Delta s_0^1 = dx^1; \Delta s_0^a = dx^a; \Delta s_q^a = ds_q^a, q = \overline{1,p}$, which shows that the local coordinates $x^i; s_1^a = \frac{dx^a}{dx^1}; \ldots; s_p^a = \frac{d^p x^a}{(dx^1)^p}$ in the space of tangent p-elements $S^p(V_n)$ are the integrals of the completely integrable system of forms $\Delta s_0^i; \Delta s_1^a; \ldots; \Delta s_p^a$.

Fixing the relative coordinates

$$v_1^i = \delta_1^i; v_2^i = \cdots = v_p^i = 0 \tag{9}$$

of p-velocity v_x^p in the frame $r_x^p \in H^p(V_n)$ we obtain:

$$s_1^a = \frac{v_1^a}{v_1^1} = 0; s_2^a = \frac{\Delta s_1^a}{v_1^1 dt} = 0; \ldots; s_p^a = \frac{\Delta s_{p-1}^a}{v_1^1 dt} = 0.$$

The canonical forms of the space $S^p(V_n)$ are identified with the following structural forms of p-frame's bundle $H^p(V_n) : \Delta s_0^i = \omega^i; \Delta s_1^a = \omega_{(1)}^a; \ldots; \Delta s_p^a = \omega_{(p)}^a$, as it is clear from (8).

We summarize our results as follows:

Proposition 3. *As a result of (9) a set of q-frames* ($q = 1, 2, \ldots, p$) *is connected with each q-element of tangency for which* $s^a_1 = \cdots = s^a_q = 0$ *with respect to these frames that generates the canonical mapping* $\kappa : H^q(V_n, D^q_n) \longrightarrow S^q(V_n)$ *stratifying* $H^q(V_n)$ *in the indicated sets of frames.*

By virtue of fibre map κ structural equations (4) of cross-section (2) are lifted into the equations of the complete preimage $\kappa^{-1}(s(S^{p-1}(V_n)))$. The obtained lifting represents the maximum submanifold of frames in $H^p(V_n)$ with respect to which only the p-elements of tangency belonging to the cross-section have zero coordinates.

Definition 1. *The curve in the base* V_n, *the differential lifting of which (natural lifting) into the space* $S^p(V_n)$ *belongs to the cross-section* $s(S^{p-1}(V_n))$, *should be called the integral curve of differential system (1).*

Thus, from (8) and Proposition 3 we come to the following statement:

Proposition 4. *The natural lifting of the integral curves of system (1) and only they are determined with respect to the frame manifold* $\kappa^{-1}(s(S^{p-1}(V_n))$ *by the following differential equation.*

$$\omega^a = \omega^a_{(1)} = \cdots = \omega^a_{(p-2)} = \omega^a_{(p-1)} = 0. \tag{10}$$

Taking into account that $\omega^a_{(1)}, \ldots, \omega^a_{(p-1)}$ are the connection forms we have the following result:

Theorem 2. *The integral curves of system (1) and only they are geodesics of the reductive connection associated with this system as the generalized autoparallel lines along which the (p-1) - order tangent element is carried in parallel.*

4. Systems of simple type characterizing the reductive connection with the zero curvature-torsion tensor

Definition 2. *All ordinary differential systems admitting the reduction to the form*

$$\frac{d^p x^a}{(dx^1)^p} = 0 \tag{11}$$

in some local coordinates would be called the systems of the simple type.

It is seemed interesting to characterize this type of systems in the terms of the curvature-torsion object of reductive Cartan connection. As it was noted earlier As it was noticed earlier in consequence of vanishing the curvature-torsion tensor, structural equations (6) of the connection are equal to Maurer-Cartan equations (7) of fundamental group G^{p-1}_n. In this case the connection forms

$$\omega^1, \omega^a, \omega^a_{(1)}, \ldots, \omega^a_{(p-1)}, \omega^1_1, \omega^1_{11}, \omega^a_b$$

may be reduced to the invariant forms

$$\Theta^1, \Theta^a, \Theta^a_{(1)}, \ldots, \Theta^a_{(p-1)}, \Theta^1_1, \Theta^1_{11}, \Theta^a_b$$

of G_n^{p-1} by transforming the variables being contained in. The solution of this problem is reduced to the solution of the completely integrable system $\Theta = \omega$. Besides, by virtue of Proposition 4, Pfaff equations $\omega^a = \omega^a_{(1)} = \cdots = \omega^a_{(p-1)} = 0$ being the differential equations of the integral curves of system (1) may be written in the terms of the differential forms of G_n^{p-1} : $\Theta^a = \Theta^a_{(1)} = \cdots = \Theta^a_{(p-1)} = 0$ which means the invariance of the integral curves and system (1) over G_n^{p-1}.

It is proved [8] the justice of follows:

Proposition 5. *System (1) is G_n^{p-1} - group-invariant if and only if it is the system of simple type (11).*

Thus, vanishing the curvature-torsion tensor of the reductive connection implies the invariance of system (1) over G_n^{p-1} and hence, by virtue of Proposition 5, it is reduced to simple type (11) with a set of G_n^{p-1} - group-invariant integral curves

$$x^a = \sum_{q=0}^{p-1} \frac{1}{q!} C^a_{(q)} (x^1)^q. \tag{12}$$

According to Theorem 2 parabolas (12) are the geodesics in the connection associated with system (11).

Owing to the uniqueness of the reduction of $H^p(V_n)$ resulting in associating the principal bundle $Q^{G_n^{p-1}}$ with differential system (1) we obtain the justice of the inverse statement.

So, we may state our result in the following theorem:

Theorem 3. *Vanishing the curvature-torsion tensor of the reductive connection associated with system (1) is equivalent to the reduction of system (1) to simple type (11).*

Without going into detail let's note that the curvature-torsion tensor of the reductive connection associated with system (1) may be basic for the classification of differential systems.

It is also proved [5],[8] the justice of the following

Theorem 4. *Fundamental group G_n^{p-1} of reductive Cartan connection is represented in R^n by the following transformations:*

$$\widetilde{x}^1 = \frac{y^1_1 x^1 + y^1}{y^1_{11} x^1 + y}; \widetilde{x}^a = \frac{y^a_b x^b + \sum_{q=0}^{p-1} \frac{1}{q!} y^a_{\underbrace{1\ldots 1}_{q}} (x^1)^q}{(y^1_{11} x^1 + y)^{p-1}}.$$

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