DIFFERENTIAL GEOMETRY AND APPLICATIONS
Proc. Conf., Aug. 28 – Sept. 1, 1995, Brno, Czech Republic
Masaryk University, Brno 1996, 275–284

GENERALIZED LAPLACE INVARIANTS AND CLASSICAL INTEGRATION METHODS FOR SECOND ORDER SCALAR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS IN THE PLANE

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ABSTRACT. In this paper we announce several new results concerning classical integration methods for second order scalar hyperbolic partial differential equations in the plane. We find that the vanishing of the generalized Laplace invariants is both necessary and sufficient for the equation to be Darboux integrable. We invariantly characterize the various general cases of Darboux integrability due to Goursat. We also find necessary and sufficient conditions for an equation to admit a general or a complete intermediate integral. Our methods are based upon the introduction of an adapted coframe for the prolonged equation manifold. This coframe is constructed by generalizing the classical Laplace transform used to integrate certain linear equations in the plane.

1. Introduction

In the 19-th century vast interest of certain mathematicians in finding exact solutions to differential equations resulted in the development of methods such as the method of Lagrange, Monge, Boole and Ampére. G. Darboux [3] generalized the method of Monge (known as the method of intermediate integrals) to obtain the most powerful method for exact integration of partial differential equations known in the last century. The idea behind the method of Darboux in the case of a second order p.d.e. in two dependent and one independent variables is to find two new equations (of any order) each of them involving an arbitrary function, such that this system of three equations, after certain prolongations, becomes Frobenius. Such a system can be integrateted by the methods of o.d.e. to obtain a general solution of the original equation in finite terms depending on two arbitrary functions and its derivatives. A long standing problem first raised by S. Lie [13]

1991 *Mathematics Subject Classification.* 58G16.
Key words and phrases. Darboux integrable, exactly integrable, Laplace invariants, intermediate integrals.
This paper is in final form and no version of it will be submitted for publication elsewhere.

is to classify, up to the class of classical contact transformations, all second order scalar p.d.e.'s in the plane that are integrable by the method of Darboux. As proposed by P. J. Vassiliou [15], we will refer to this problem as the Darboux-Goursat-Vessiot (DGV) problem. Special cases of the DGV problem were studied by S. Lie [13], E. Goursat [6], and others in the second half of the 19-th century. S. Lie proved that the only Darboux integrable equation of type $u_{xy} = f(u)$ is the Liouville equation

$$u_{xy} = ae^{bu}, \tag{1.1}$$

where a and b are constants. In the end of the last century E. Goursat [6] succeeded in classifying hyperbolic equations of type

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} = f(x,y,u,u_x,u_y)$$

with respect to the class of fiber preserving transformations of type $\overline{x} = \overline{x}(x)$, $\overline{y} = \overline{y}(y)$, and $\overline{u} = \overline{u}(x,y,u)$, that are Darboux integrable at order two. He found a representative for each equivalence class and then integrated this representative to obtain a general solution. A substantial contribution to DGV problem was made by E. Vessiot 40 years later, who successfully applied his theory of differential systems, developed in [17], to the explicit integration of nonlinear wave equations

$$u_{xy} = f(x,y,u,u_x,u_y). \tag{1.2}$$

In his papers [18], [19] Vessiot was able to classify all equations of type (1.2) that admit general integrals of the form

$$u = F^0(\xi,\xi',\dots,\xi^{(p)},\eta,\eta',\dots,\eta^{(q)},\alpha,\beta), \tag{1.3}$$

$$x = F^1(\xi,\xi',\dots,\xi^{(p)},\alpha), \quad y = F^2(\eta,\eta',\dots,\eta^{(q)},\beta) \tag{1.4}$$

where $\xi(\alpha)$ is an arbitrary function of α and $\eta(\beta)$ is an arbitrary function of β. Vessiot also gave algorithms for constructing such solutions. Recently, there was a renewed interest in the method of Darboux that was studied in [1], [2], [4], [10], [12], [15], and [16].

Goursat also proved that a linear, scalar, hyperbolic partial differential equation in the plane

$$u_{xy} + a(x,y)u_x + b(x,y)u_y + c_(x,y)u = 0 \tag{1.5}$$

is Darboux integrable if and only if the associated two sequences of Laplace invariants are finite. I. Anderson and N. Kamran [1] introduced generalized Laplace invariants by constructing an adapted coframe for the generalized Laplace transform. In [1] it was proved that the finiteness of the two sequences of generalized Laplace invariants is a necessary condition for a hyperbolic equation to be integrable by the method of Darboux. Recently Sokolov and Ziber [14], succeded in proving the converse for the class of nonlinear wave equations (1.2). In this paper we outline the proof of the converse in general, i.e. we conclude that *a second order, scalar, hyperbolic partial differential equation in the plane is Darboux integrable if and only if the two associated sequences of generalized Laplace invariants are finite.* This fully generalizes the classical result of Goursat. We also provide a contact invariant characterization of the various cases of Darboux integrability due to Goursat [8]; in particular we obtain a characterization of equations that admit general and complete intermediate integrals.

This research was partly supported by NFS grant DMS 94-03783 and by a research assistantship from the Department of Mathematics at Utah State University. I would like to thank my advisor Prof. Ian Anderson for his guidance throughout the writing of this paper and for many valuable comments and suggestions.

2. Preliminaries and Definitions

We quickly review the construction of the Laplace adapted coframe introduced in [1]. Consider the trivial bundle $E = \{\pi : E \to M\}$, with local coordinates $\pi : (x, y, u) \to (x, y)$ and let $\pi^\infty_M : J^\infty(E) \to M$ be the infinite jet bundle of the local sections of π. The second order equation

$$F(x, y, u, p, q, r, s, t) = 0, \tag{2.1}$$

where x and y are independent variables, u denotes a dependent variable and for the derivatives of u we use the Monge notation $u_x = p, u_y = q, u_{xx} = r, u_{xy} = s, u_{yy} = t$, defines the 7 dimensional equation manifold $\mathcal{R}^2 \hookrightarrow J^2(E)$. The equation (2.1) together with all its differential consequences, defines the infinitely prolonged equation manifold $\mathcal{R}^\infty \hookrightarrow J^\infty(E)$. The *characteristic equation* for (2.1) is

$$\frac{\partial F}{\partial r}\lambda^2 - \frac{\partial F}{\partial s}\lambda\mu + \frac{\partial F}{\partial t}\mu^2 = 0.$$

In this paper we restrict ourselves to the case of *hyperbolic equations*, i.e. to the equations for which

$$\left(\frac{\partial F}{\partial s}\right)^2 > \frac{\partial F}{\partial r}\frac{\partial F}{\partial t}$$

at any point of the equation manifold $\mathcal{R}^2$. Hence the characteristic equation has a pair of non-proportional real roots $(\lambda, \mu) = (m_x, m_y)$ on $\mathcal{R}^2$ and $(\lambda, \mu) = (n_x, n_y)$ which define the *characteristic vector fields*

$$X = m_x D_x + m_y D_y, \qquad \text{and} \qquad Y = n_x D_x + n_y D_y.$$

The Lie bracket of total vector fields is again a total vector field. We define the functions P and Q on $\mathcal{R}^\infty$ by

$$[X, Y] = PX + QY.$$

We denote the horizontal forms dual to the characteristic vector fields X and Y by σ and τ, respectively. On the infinitely prolonged equation manifold $\mathcal{R}^\infty$ the contact forms

$$\theta = du - pdx - qdy, \quad \theta_x = dp - rdx - sdy, \quad \theta_y = dq - sdx - tdy, \ \ldots$$

are related by the *universal linearization*

$$\frac{\partial F}{\partial r}\theta_{xx} + \frac{\partial F}{\partial s}\theta_{xy} + \frac{\partial F}{\partial t}\theta_{yy} + \frac{\partial F}{\partial p}\theta_x + \frac{\partial F}{\partial q}\theta_y + \frac{\partial F}{\partial u}\theta = 0.$$

We let $\Theta = \varrho\theta$ for any smooth function ϱ on $\mathcal{R}^\infty$, and we rewrite this equation using the characteristic vector fields as

$$XY(\Theta) + AX(\Theta) + BY(\Theta) + C(\Theta) = 0, \tag{2.2}$$

and, equivalently, as

$$YX(\Theta) + DX(\Theta) + EY(\Theta) + G(\Theta) = 0. \tag{2.3}$$

Explicit formulas for the coefficients $A, B, C, \dots,$ are given in [1]. The Laplace adapted coframe on $\mathcal{R}^\infty$ is constructed by successive applications of the classical Laplace transform to (2.2) and (2.3). The first elements of this coframe are defined by

$$\eta_1 = Y(\Theta) + A\Theta, \quad H_0 = X(A) + AB - C,$$

and

$$\xi_1 = X(\Theta) + E\Theta, \quad K_0 = Y(E) + ED - G.$$

Then provided $H_0 \neq 0, H_1 \neq 0, \dots, H_{i-1} \neq 0$, η_i satisfies the identity

$$XY(\eta_i) + A_iX(\eta_i) + B_iY(\eta_i) + C_i(\eta_i) = 0.$$

We set

$$\eta_{i+1} = Y(\eta_i) + A_i\eta_i, \quad \text{and} \quad H_i = X(A_i) + A_iB_i - C_i.$$

This process continues until $H_p = 0$ in which case we define

$$\eta_{p+i+1} = Y(\eta_{p+i}) \quad \text{for all } i \geq 1.$$

Similarly, provided $K_0 \neq 0, K_1 \neq 0, \dots, K_{i-1} \neq 0$, ξ_i satisfies the identity

$$YX(\xi_i) + D_iX(\xi_i) + E_iY(\xi_i) + G_i(\xi_i) = 0,$$

and we set

$$\xi_{i+1} = X(\xi_i) + E_i\xi_i, \quad \text{and} \quad K_i = Y(E_i) + D_iE_i - G_i.$$

This process continues until $K_q = 0$ in which case we define

$$\xi_{q+i+1} = X(\xi_{q+i}) \quad \text{for all } i \geq 1.$$

The forms $\{\sigma, \tau, \Theta, \eta_1, \xi_1, \eta_2, \xi_2, \dots\}$ define the *Laplace adapted coframe.* The functions $H_0, K_0, H_1, K_1, \dots$ are relative invariants under the classical contact transformations and are called the *generalized Laplace invariants.* Let R'^∞ be another second order scalar hyperbolic equation in the plane and let $\Phi : \mathcal{R}^\infty \to R'^\infty$ be a classical contact transformation. Let X' and Y' be the characteristic vector fields for R'^∞ and let $\{\sigma', \tau', \Theta', \eta'_1, \xi'_1, \eta'_2, \xi'_2, \dots\}$ be the Laplace adapted coframe for R'^∞. Then with

$$X' = mX, \quad Y' = nY, \quad \text{and} \quad \Theta' = l\Theta, \tag{2.4}$$

the generalized Laplace invariants are related by $H'_i = mnH_i$ and $K'_i = mnK_i$. Note that in the case of a linear equation the generalized Laplace invariants and the Laplace invariants coincide. The following recursion formulas for the generalized Laplace invariants generalize the recursion formulas for the classical Laplace invariants.

$$H_n = H_{n-1} - XY\ln(H_0H_1\cdots H_{n-2}H_{n-1}) + QY\ln(H_0H_1\cdots H_{n-2}H_{n-1})$$
$$+H_0 - K_0 + (n+1)(Y(Q)+2PQ-X(Q)), \quad \text{and} \tag{2.5}$$
$$K_n = K_{n-1} - YX\ln(K_0K_1\cdots K_{n-2}K_{n-1}) - PX\ln(K_0K_1\cdots K_{n-2}K_{n-1})$$
$$+K_0 - H_0 + (n+1)(Y(Q)+2PQ-X(Q)), \qquad n \geq 1. \tag{2.6}$$

Definition 2.1. We say that the equation (2.1) is *Darboux integrable* if there exist four functions I_1, I_2, J_1 and J_2, such that the system

$$F = 0, \quad I_2 = \varphi(I_1) \text{ and } J_2 = \psi(J_1),$$

after a finite number of prolongations, becomes Frobenius for any choice of the functions φ and ψ.

Definition 2.2. We say that the equation (2.1) *admits a general intermediate integral* if there are two first order functions I, J, such that the system

$$F = 0, \qquad J = \varphi(I), \qquad D_xJ = D_x\varphi(I) \qquad \text{and} \qquad D_yJ = D_y\varphi(I)$$

is in involution with character 1 and Cartan integer 1 for any choice of the function φ.

Definition 2.3. We say that the equation (2.1) *admits a complete intermediate integral* if there are two second order functions I and J, such that the system

$$F = 0, \qquad I = a \qquad \text{and} \qquad J = b$$

is in involution with character 1 and Cartan integer 1 for any choice of constants a, b.

3. Characterization of Darboux integrable equations

Theorem 3.1. *A second order, scalar, hyperbolic partial differential equation in the plane is Darboux integrable if and only if both sequences of it's generalized Laplace invariants are finite.*

The necessity of the vanishing of the generalized Laplace invariants for the Darboux integrable equations was established in [1]. Sokolov and Ziber [14] proved the theorem for the special case of nonlinear wave equations (1.2).

Proof: A hyperbolic equation (2.1) is integrable by the method of Darboux if and only if there exist two functionally independent X invariant functions I_1 and I_2 on $\mathcal{R}^\infty$ and two functionally independent Y-invariant functions J_1 and J_2 on $\mathcal{R}^\infty$, i.e. $X(I_1) = X(I_2) = 0$ and $Y(J_1) = Y(J_2) = 0$. The necessity of the finiteness of the two sequences of generalized Laplace invariants now follows from the examination of the d_H-structure equations of the Laplace adapted coframe, see [1] for details. To prove the converse let $H_p = 0$, $K_q = 0$ for some integers

p and q. Then there are uniquely defined forms $\hat{\tau} \in \Omega^1(\tau, \xi_1, \Theta, \eta_1, \dots, \eta_p)$ and $\hat{\sigma} \in \Omega^1(\sigma, \eta_1, \Theta, \xi_1, \dots, \xi_q)$, such that

$$d\eta_{p+1} = \eta_{p+2} \wedge \hat{\tau} + \eta_{p+1} \wedge \eta, \qquad \text{and} \qquad d\xi_{q+1} = \xi_{q+2} \wedge \hat{\sigma} + \xi_{q+1} \wedge \xi.$$

Note that the forms $\{\hat{\tau}, \hat{\sigma}, \Theta, \eta_1, \xi_1, \eta_2, \xi_2, \dots\}$ form a basis of 1-forms on $\mathcal{R}^\infty$ and under the classical contact transformation (2.8) the forms $\hat{\sigma}, \hat{\tau}, \xi_{q+1}, \xi_{q+2}, \dots,$ $\eta_{p+1},\ \eta_{p+2}, \dots,$ transform as

$$\hat{\sigma}' = \frac{1}{m}\hat{\sigma}, \quad \hat{\tau}' = \frac{1}{n}\hat{\tau}, \quad \text{and} \quad \xi'_{q+i} = m^{q+i}\xi_{q+i}, \quad \eta'_{p+i} = n^{p+i}\eta_{p+i} \quad \text{for} \quad i \geq 1.$$

Computing the d-structure equations we obtain

$$d\hat{\tau} \equiv 0 \mod \{\hat{\tau}, \eta_{p+1}, \eta_{p+2}\}, \quad d\eta_{p+1} \equiv 0 \mod \{\hat{\tau}, \eta_{p+1}, \eta_{p+2}\}, \tag{3.1}$$

$$d\eta_{p+i} \equiv 0 \mod \{\hat{\tau}, \eta_{p+1}, \dots, \eta_{p+i}\} \quad \text{for} \quad i \geq 2, \qquad \text{and}$$

$$d\hat{\sigma} \equiv 0 \mod \{\hat{\sigma}, \xi_{q+1}, \xi_{q+2}\}, \quad d\xi_{q+1} \equiv 0 \mod \{\hat{\sigma}, \xi_{q+1}, \xi_{q+2}\}, \tag{3.2}$$

$$d\xi_{q+i} \equiv 0 \mod \{\hat{\sigma}, \xi_{q+1}, \dots, \xi_{q+i}\} \quad \text{for} \quad i \geq 2. \quad \square$$

To state the contact invariant characterization of Goursat's general classification of hyperbolic equations, we need to introduce the following two invariants.

$$M_\tau = \frac{1}{\varrho\Delta^3}[(\frac{\partial n_y}{\partial r}n_x - \frac{\partial n_x}{\partial r}n_y)n_y^2 - (\frac{\partial n_y}{\partial s}n_x - \frac{\partial n_x}{\partial s}n_y)n_x n_y + (\frac{\partial n_y}{\partial t}n_x - \frac{\partial n_x}{\partial t}n_y)n_x^2], \tag{3.3}$$

and

$$M_\sigma = -\frac{1}{\varrho\Delta^3}[(\frac{\partial m_y}{\partial r}m_x - \frac{\partial m_x}{\partial r}m_y)m_y^2 - (\frac{\partial m_y}{\partial s}m_x - \frac{\partial m_x}{\partial s}m_y)m_x m_y + (\frac{\partial m_y}{\partial t}m_x - \frac{\partial m_x}{\partial t}m_y)m_x^2], \tag{3.4}$$

where $\Delta = m_x n_y - m_y n_x$. The transformation laws of M_σ and M_τ under the classical contact transformations (2.4) are given by

$$M'_\sigma = \frac{n}{m^3 l}M_\sigma, \quad \text{and} \quad M'_\tau = \frac{m}{n^3 l}M_\tau.$$

Recall that a *Monge -Ampére* (MA) *equation* is an equation of type

$$E(s^2 - rt) + Ar + 2Bs + Ct + D = 0,$$

where A, B, C, D, E are functions of $x, y, u, p,$ and q. *The vanishing of M_σ and M_τ is a necessary and a sufficient condition for an equation to be* MA [11]. We therefore propose to call M_σ and M_τ, MA *invariants,* and the equation for which one of the the invariants M_σ or M_τ vanishes, a *semi* MA *equation.* If the hyperbolic equation (2.1) is of the form

$$r = f(x, y, u, p, q, s, t),$$

then the characteristic vector fields can be chosen to be $X = D_x + mD_y$ and $Y = D_x + nD_y$ where m and n are the two distinct second order functions satisfying the equation

$$\lambda^2 + \frac{\partial f}{\partial s}\lambda - \frac{\partial f}{\partial t} = 0.$$

The formulas for M_σ and M_τ now become

$$M_\sigma = \frac{1}{(n-m)^3}\left(\frac{\partial n}{\partial t} - n\frac{\partial n}{\partial s}\right), \quad \text{and} \quad M_\tau = \frac{1}{(m-n)^3}\left(\frac{\partial m}{\partial t} - m\frac{\partial m}{\partial s}\right). \tag{3.5}$$

The formulas for H_0 and K_0 are

$$H_0 = (m-n)^3 M_\sigma (u_{xyyy} + m u_{yyyy}) + \{\text{ function of order} \leq 3\}, \quad \text{and} \tag{3.6}$$

$$K_0 = (n-m)^3 M_\tau (u_{xyyy} + n u_{yyyy}) + \{\text{ function of order} \leq 3\}. \tag{3.7}$$

By refining the d-structure equations (3.1) and (3.2) we recover the set of contact invariant conditions that characterize the general classification of Darboux integrable equations which was undertaken by E.Goursat [6] in the end of the last century. For the proof see [12].

THEOREM 3.2. *Let the equation (2.1) be hyperbolic. Let $H_p = 0$.*
(i) If $p \geq 2$ or $p = 1$ and $M_\tau = 0$, then there is one functionally independent X invariant function of order $\leq p$ and one X invariant functionally independent function of order $p+1$.
(ii) If $p = 1$ and $M_\tau \neq 0$, then there are no X invariant functions of order one and there are two functionally independent X invariant functions of order two.
(iii) If $p = 0$ and $M_\tau \neq 0$, then there are no X invariant functions of order one and there are three functionally independent X invariant functions of order two.
(iv) If $p = 0$ and $M_\tau = 0$, then there are two functionally independent X invariant functions of order ≤ 1.

Note that if there are at least two functionally independent X invariant functions of order $< k$ then there exists exactly one functionally independent X invariant function of order k.
We have the following characterization of equations that admit a general or a complete intermediate integrals [11].

THEOREM 3.3. *The following statements hold true:*
(i) A second order, scalar, hyperbolic partial differential equation in the plane admits a general intermediate integral if and only if $H_0 = M_\tau = 0$ or $K_0 = M_\sigma = 0$.
(ii) A second order, scalar, hyperbolic partial differential equation in the plane admits a complete intermediate integral if and only if $H_0 = 0$ or $K_0 = 0$.

EXAMPLE 3.1. The characteristic vector fields of the MA equation

$$p^2 q(s^2 - rt) + 2pqyr + (p^2 y + qx)s + xpt - xy = 0 \tag{3.8}$$

can be chosen as

$$X = p(2y - pt)D_x + (p^2 s + x)D_y, \quad \text{and} \quad Y = q(2y - pt)D_x + p(qs + y)D_y.$$

Since $H_0 = 0$, then we conclude that (3.8) admits a general intermediate integral. The X invariant functions are

$$I_1 = pq - \frac{1}{2}y^2, \quad \text{and} \quad I_2 = p^2q - \frac{1}{2}x^2,$$

and the integral is given by $I_2 = \varphi(I_1)$.

Example 3.2. The characteristic vector fields of the hyperbolic semi MA equation

$$rs - p = 0 \tag{3.9}$$

are given by $X = sD_x + rD_y$ and $Y = D_x$. The generalized Laplace invariants and the semi MA invariants are given by $H_0 = 0$ and $K_0, K_1, K_2 \neq 0, K_3 = 0$ and $M_\sigma = 0, M_\tau \neq 0$. Therefore by Theorem 3.3 the equation (3.9) admits a complete intermediate integral but not a general intermediate integral. The X invariant functions are

$$I_1 = s - x, \quad I_2 = y - \frac{p}{s}, \quad \text{and} \quad I_3 = \frac{s^2}{p}. \tag{3.10}$$

Notice also that

$$\frac{Y(I_1)}{Y(I_2)} = Y(I_3).$$

On the equation manifold of (3.9) the X invariant functions I_1 and I_2 satisfy the relation

$$(I_1 + x)(y - I_2) - p = 0. \tag{3.11}$$

If we substitute from (3.10) into (3.11) we recover (3.9).

Example 3.3. Consider the Liouville equation (1.1) The generalized Laplace invariants are $H_0 = K_0 = abe^{bu}$, and $H_1 = K_1 = 0$. Hence the Liouville equation admits neither general nor intermediate integrals but it is Darboux integrable at the order two.

Example 3.4. Consider a hyperbolic quasi-linear equation

$$r = t + \frac{2}{x}p \tag{3.12}$$

with characteristic vector fields $X = D_x + D_y$ and $Y = D_x - D_y$. Computing the generalized Laplace invariants we obtain $H_0 \neq 0$, $K_0 \neq 0$ and $H_1 = K_1 = 0$. The X and Y invariant functions are

$$I_1 = x - y, \quad I_2 = \frac{1}{x}(t - s) + \frac{1}{p}, \quad \text{and} \quad J_1 = x + y, \quad J_2 = \frac{1}{x}(t + s) + \frac{1}{p}.$$

It is now a simple exercise to check that the equation (3.12) together with the equations $I_2 = \varphi(I_1)$ and $J_2 = \psi(J_1)$ constitute a Frobenius system. Integrating we obtain the general solution of the equation (3.12)

$$u = \varphi(x + y) + \psi(x - y) - x(\varphi'(x + y) - \psi'(x - y)),$$

where φ and ψ are arbitrary functions.

EXAMPLE 3.5. Consider the F-Gordon equation $s = u$. A simple computation using the recursion formulas for the generalized Laplace invariants (2.5) and (2,6) shows that $H_i = K_i = 1$ for all $i = 0, 1, 2, \ldots$. Hence we conclude that the equation $s = u$ is not Darboux integrable. Note that this equation is integrable by the method of Fourier.

From the Theorem 3.3. it immediately follows that *every* MA *equation that admits a complete intermediate integral also admits a general intermediate integral.* From the formulas (3.6) and (3.7) we observe a well known fact that *every equation that admits a general intermediate integral is* MA. We conclude with two further applications of the generalized Laplace invariants.

THEOREM 3.4. *A second order, scalar hyperbolic partial differential equation in the plane is contact equivalent to the wave equation $s = 0$ if and only if $H_0 = K_0 = 0$.*

Proof: The theorem follows immediately from Theorem 3.3 and the result of S. Lie : "if the equation admits two X invariant functions of order ≤ 1, and two Y invariant functions of order ≤ 1 then it is contact equivalent to the wave equation".

From the formulas (3.6) and (3.7) we easily deduce that if $H_0 = K_0$ then the equation is MA. See [1] for a special case of the following theorem.

THEOREM 3.5. *A second order, scalar hyperbolic partial differential equation in the plane admits a variational multiplier if and only if $H_0 = K_0$.*

For the proof see [11].

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