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$F(3,1)$-STRUCTURE ON THE LAGRANGIAN SPACE AND INVARIANT SUBSPACES

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Abstract. If almost product structure P on the tangent space $T(E) = T_V(E) \oplus T_H(E)$ of Lagrangian $2n$ dimensional manifold E is defined, and if $f_v(3,1)$-structure on $T_V(E)$ is defined, then $f_h(3,1)$-structure on $T_H(E)$ are defined in the natural way. We can define $F(3,1)$-structure on $T(E)$. The condition is given for the reduction of the structural group of such manifolds.

Two cases of invariant submanifolds are considered. In the first case we get an induced almost complex structure, and in the second case we obtain an induced $f(3,1)$-structure on the invariant submanifold.

1. Introduction

Let $\mathcal{M}$ be an n-dimensional and E $2n$-dimensional differentiable manifold and let $\eta = (E, \pi, \mathcal{M})$ be vector bundles and $\pi E = \mathcal{M}$. The differential structures (U, ϕ, R^{2n}) are vector charts of the vector bundles η. Hence the canonical coordinates on $\pi^{-1}(U)$ are $(x^1, \dots, x^n, y^1, \dots, y^n) = (x^i, y^a)$, $i = 1, 2, \dots, n$; $a = 1, \dots, n$. Transformation maps on E are $x^{i'} = x^{i'}(x^1, x^2, \dots, x^n)$,
$y^{a'} = M_a^{a'}(x^1, \dots, x^n) y^a$ and $\operatorname{rank}\left[\frac{\partial x^{i'}}{\partial x^i}\right] = \operatorname{rank}\left[\frac{\partial y^{a'}}{\partial y^a}\right] = \operatorname{rank} M_a^{a'} = n$.

The inverse transformations are

$$x^i = x^i(x^{1'}, x^{2'}, \dots, x^{n'}),\ y^a = M_{a'}^a(x^{i'}, \dots, x^{n'}) y^{a'},\ \text{where } M_{a'}^a M_b^{a'} = \delta_b^a.$$

The local natural bases of the tangent space $T(E)$ are $\{\partial_i, \partial_a\}$

$$\partial_a = \frac{\partial}{\partial y^a} = M_a^{a'}(x^i)\partial_{a'},\ \ \partial_i = \frac{\partial}{\partial x^i} = \frac{\partial x^{i'}}{\partial x^i}\partial_{i'} + (\partial_i M_b^{a'}(x^i)) y^b \partial_{a'}.$$

The nonlinear connection on E is distribution $N : u \in E \to N_u \subset T_u(E)$ which is supplementary to the distribution V,

$$T_u(E) = N_u \oplus V_u, \quad \forall_u \in E. \tag{1}$$

They are localy determined by $\delta_i = \partial_i - N_i^a \partial_a$. The local bases adapted to decompositions in (1.1) is $\{\delta_i, \partial_a\}$.

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It is easy to prove [4] that on $\{\delta_i, \partial_a\}$

$$\delta_{i'} = \delta_i \frac{\partial x^i}{\partial x^{i'}}, \quad \partial_{a'} = \frac{\partial y^a}{\partial y^{a'}} \partial_a.$$

The subspace of $T(E)$ spaned by $\{\delta_i\}$ will be denoted by $T_H(E)$ and the subspace spaned by $\{\partial_a\}$ will be denoted by $T_V(E)$, $T(E) = T_H(E) \oplus T_V(E)$, $\dim T_H(E) = n = \dim T_V(E)$.

Definition 1.1. If the Riemannian metrical structure on $T(E)$ is given by $G = g_{ij}(x^i, y^a) dx^i \otimes dx^j + g_{ab}(x^i, y^a) \delta y^a \otimes \delta y^b$ where $g_{ij}(x^i, y^a) = g_{ij}(x^i)$, $g_{ab} = \frac{1}{2} \partial_a \partial_b L(x^i, y^a)$ and $L(x^i, y^a)$ is a Lagrange function, then such a space we call Lagrangian space.

Let $X \in T(E)$, then $X = X^i \delta_i + \bar{X}^a \partial_a$ and the automorphism $P : \mathcal{X}(T(E)) \to \mathcal{X}(T(E))$ defined by $PX = \bar{X}^i \delta_i + X^a \partial_a$ is the natural almost product structure on $T(E)$ i.e, $P^2 = I$. If we denote by v and h the projection morphism of $T(E)$ to $T_V(E)$ and $T_H(E)$ respectively, we have $P \circ h = v \circ P$.

2. $f(3,1)$-structures

Definition 2.1. We call Lagrange vertical $f_v(3,1)$-structure of rank r on $T_V(E)$ a non-null tensor field f_v of type (1,1) and of class C^∞ such that $f_v^3 + f_v = 0$, and rank $f_v = r$, where r is constant everywhere.

Definition 2.2. We call Lagrange horizontal $f_h(3,1)$-structure on $T_H(E)$ a non-null tensor field f_h on $T_H(E)$ of type (1,1) of class C^∞ satisfying $f_h^3 + f_h = 0$, rank $f_h = r$, where r is constant everywhere.

An $F(3,1)$-structure on $T(E)$ is a non-null tensor field F of type $\binom{11}{11}$ such that $F^3 + F = 0$, rank $F = 2r$ =const.

For our study it is very convenient to consider f_v or f_h as morphism of vector bundles [2], [3]

$$f_v : \mathcal{X}(T_V(E)) \to \mathcal{X}(T_V(E)), \quad f_h : \mathcal{X}(T_H(E)) \to \mathcal{X}(T_H(E)).$$

Let f_v be a Lagrange vertical $f_v(3,1)$-structure of rank r. We define the morphisms $l_v = -f_v^2$ and $m_v = f_v^2 + I_{T_V(E)}$, where $I_{T_V(E)}$ denotes the identity morphism on $T_V(E)$.

It is clear that $l_v + m_v = I$. Also we have $l_v m_v = m_v l_v = -f_v^4 - f_v^2 = -f_v(f_v^3 + f_v^1) = 0$, $m_v^2 = m_v$, $l_v^2 = l_v$.

Hence the morphisms l_v, m_v applied to the $\mathcal{X}(T_V(E))$ are complementaly projection morphisms, then there exist complementary distributions L_v and M_v corresponding to the projection morphisms l_v and m_v respectively such that dim $L_v = r$ and dim $M_v = n - r$. It is easily to see that

$$\begin{aligned} &l_v f_v = f_v l_v = f_v, \; m_v f_v = f_v m_v = 0, \\ &f_v^2 l_v = l_v f_v^2 = -I, \; f_v^2 m_v = 0, \end{aligned} \tag{1}$$

Proposition 2.1. If a Lagrange $f_v(3,1)$-structure of rank r defined on $T_V(E)$, then the horizontal $f_h(3,1)$-structure of rank r is defined on $T_H(E)$ by the natural almost product structure of $T(E)$.

Proof. If we put

$$f_h X = P f_v P X, \quad \forall X \in T_H(E) \tag{2}$$

it is easy to see that $f_h^3 X = P f_v^3 P X$ and $f_h^3 + f_h = 0$, and rank $f_h = r$.

Proposition 2.2. If a Lagrange $f_v(3,1)$-structure of rank r is defined on $T_V(E)$, then an $F(3,1)$-structure is defined on $T(E)$ by the natural almost product structure of $T(E)$.

Proof. We put

$$F = f_h h + f_v v, \tag{3}$$

where f_h, is defined by (2.2), and h, v are the projection morphisms of $T(E)$ to $T_H(E)$ and $T_V(E)$. Then it is easy to check that $F^2 = f_h^2 h + f_v^2 v$, $F^3 = f_h^3 h + f_v^3 v$. Thus $F^3 + F = 0$. It is clear that
rank $F = 2r$.

If l_h, m_h are complementary projection morphisms of the horizontal $f_h(3,1)$-structure, which is defined by the natural almost product structure of $T(E)$, we have

$$\begin{aligned} &l_h X = -f_h^2 X = -P f_v^2 P X = P l_v P X, \forall X \in T_H(E) \\ &m_h X = (f_h^2 + I_{T_H(E)}) X = P f_v^2 P X + P I_{T_V(E)} P X = P m_v P X, \\ &m_h X = P m_v P X, \ \forall X \in T_H(E). \end{aligned}$$

If l, m are complementary projection morphism of the $F(3,1)$-structure on $T(E)$, then we have

$$\begin{aligned} l &= -F^2 = -f_h^2 h - f_v^2 v = l_h h + l_v v \\ m &= F^2 + I_{T(E)} = f_h^2 h + f_v^2 v + I_{T_H(E)} h + I_{T_V(E)} v = m_h h + m_v v. \end{aligned} \tag{4}$$

Thus, if there is given a Lagrange $f_v(3,1)$-structure on $T_V(E)$ of rank r, then there exist complementary distributions L_h, M_h of $T_H(E)$, corresponding to the morphisms l_h, m_h such that $L_h = PL_v$, $M_h = PM_v$. Thus we have the decompositions

$$T(E) = T_H(E) \oplus T_V(E) = PL_v \oplus PM_v \oplus L_v \oplus M_v. \tag{5}$$

If L, M denote complementary distributions corresponding to the morphisms l, m respectively then from (2.4) and (2.5) we have $L = PL_v \oplus L_v$, $M = PM_v \oplus M_v$.

Let $\bar{g}_v$ is a pseudo-Riemannian metric tensor, which is symmetric, bilinear and non-degenerate on $T_V(E)$

$$\bar{g}_v : \mathcal{X}(T_V(E)) \times \mathcal{X}(T_V(E)) \to \mathcal{F}(T(E)).$$

(For examples $\bar{g}_v$ can be the vertical part of Lagrange metric structure).

The mapping $a_v : \mathcal{X}(T_V(E)) \times \mathcal{X}(T_V(E)) \to \mathcal{F}(T(E))$ which is defined by

$$a_v(X,Y) = \frac{1}{2}[\bar{g}(l_v X, l_v Y) + \bar{g}(m_v X, m_v Y)] \ \ \forall X, Y \in \mathcal{X}(T_V(E))$$

is a pseudo-Riemannian structure on $T(E)$ such that $a_v(X,Y) = 0$, $\forall X \in \mathcal{X}(L_v)$, $Y \in \mathcal{X}(M_v)$.

Theorem 2.1. If a Lagrange $f_v(3,1)$-structure of rank r is defined on $T_V(E)$ then there exist a pseudo-Riemannian structure of $T_V(E)$ with respect to which the complementary distributions L_v and M_v are orthogonal and f_v is an isometry on $T_V(E)$.

Proof. If we put $g_v(X,Y) = \frac{1}{2}[a_v(X,Y) + a_v(f_vX, f_vY)]$ it is easy to see that $g_v(X,Y) = 0 \quad \forall X \in \mathcal{X}(L_v), \quad Y \in \mathcal{X}(M_v)$. Using (2.1) we get $g_v(f_vX, f_vY) = \frac{1}{2}[a_v(f_vX, f_vY) + a_v(X,Y)]$. Thus f_v is an isometry with respect to g_v.

Let $X \in \mathcal{X}(L_v)$ then $f_vX,\ f_v^2X \in \mathcal{X}(L_v)$ and $g_v(X, f_vX) = g_v(f_vX, f_v^2X) = -g_v(f_vX, X)$.

Consequently $g_v(X, f_vX) = g_v(f_vX, f_v^2X) = 0$ and $r = 2m$.

Thus we can chose in $r = 2m$ mutualy orthogonal unit vector fields $\mathcal{X}(L_v)$ such that

$$\begin{aligned} f_v(X_a) &= X_{\alpha+m} && \alpha = 1,2,\dots,m, \\ f_v(X_\alpha) &= -X_{-m+\alpha}, && \alpha = m+1,\dots,2m. \end{aligned}$$

An adapted frame of the Lagrange $f_v(3,1)$-structure on $T_V(E)$ is the orthogonal frame $R = \{X_\alpha, X_\beta\}$, where X_β is an orthogonal frame of $\mathcal{X}(M_v)$.

Let $\bar{R} = \{\bar{X}_\alpha, \bar{X}_\beta\}$ be another adapted frame of the Lagrange $f_v(3,1)$-structure, and $\bar{R} = AR$, then orthogonal matrix A is an element of the group $U_{(m)} \times O_{(n-2m)}$.

Theorem 2.2. A necessary and suficient condition for $T_V(E)$ to admit Lagrange $f_v(3,1)$-structure of rank r is that $r = 2m$ and the structure group of the tangent bundle of the manifold can be reduced to the group $U_{(m)} \times O_{(n-2m)}$.

We can define a metric structure g_h on $T_H(E)$: $g_h(X,Y) = g_v(PX, PY)$, $\forall X, Y \in \mathcal{X}(T_H(E))$. Using (2.5) the distributions L_h, M_h are orthogonal with respect to g_h and the horizontal $f_h(3,1)$-structure which is define by $f_hX = Pf_vPX$, $\forall X \in \mathcal{X}(T_H(E))$ is an isometry on $T_H(E)$ with respect to g_h.

We can also define a metric tensor G on $T(E)$.

$$G(X,Y) = g_h(X,Y)h + g_v(X,Y)v. \tag{6}$$

The distributions L, M are orthogonal with respect to G and the $F(3,1)$-structure which is defined by $FX = f_hh + f_vv$, $X \in T(E)$ is an isometry on $T(E)$ with respect to G.

Proposition 2.3. If $\{X_\alpha, X_\beta\}$ is an adapted frame of a given Lagrange $f_v(3,1)$-structure f_v on $T_V(E)$ with respect to g_v, then the frame $\{PX_\alpha, PX_\beta\}$ is an adapted frame of the horizontal $f_h(3,1)$-structure with respect to g_h.

It is clear that the frame $\{PX_\alpha, PX_\beta, X_\alpha, X_\beta\}$ is adapted frame to the decomposition $T(E) = L_h \oplus M_h \oplus L_v \oplus M_v$.

Theorem 2.3. If a Lagrange $f_v(3,1)$-structure is defined on $T_V(E)$ with pseudo-Riemannian structure g_v, then the structure group of the tangent bundle on $T(E)$ can be reduced to $U_{(m)} \times O_{(n-2m)} \times U_{(m)} \times O_{(n-2m)}$.

3. Invariant Submanifolds

Let $\mathcal{N}^p$ be an p-dimensional C^∞ manifold imbedded, differentially as a submanifold in a $2n$-dimensional C^∞ manifold E^{2n} with metric structure G and $F(3,1)$-structure, as in Chapter 2. Let Φ be an imbedding map $\Phi : \mathcal{N}^p \to E^{2n}$, and $\Phi_*(\equiv B)$ the Jacobian map of Φ i.e.
$B : T(\mathcal{N}^p) \to T(E)$. Denoting by $T(\mathcal{N}, E)$ the set of all vectors tangent to the submanifold $\Phi(\mathcal{N}^p)$. It is known that $B : T(\mathcal{N}^p) \to T(\mathcal{N}, E)$ is an isomorphism. Take the C^∞ vector field $\bar{X} = B\tilde{X}$ and defined along $\Phi(\mathcal{N}^p)$ which is tangential to $\Phi(\mathcal{N}^p)$ and let X be the local C^∞-extension of $\bar{X}$. We define the induced metric $\tilde{g}$ on $\mathcal{N}^m$ as follows

$$\tilde{g}(\tilde{X}, \tilde{Y}) = G(B\tilde{X}, B\tilde{Y}) \text{ for all } \tilde{X}, \tilde{Y} \in T(\mathcal{N}^p), \tag{1}$$

where G is the Riemannian metric in E. It can be easily verified that $\tilde{g}$ is Riemanian metric in $\mathcal{N}^p$.

Definition 3.1. $\mathcal{N}^p$ is said to be an invariant submanifold of E if the tangent space $T_u(\Phi(\mathcal{N}^p))$ of $\Phi(\mathcal{N}^p)$ is invariant by the linear mapping F at each point of $\Phi(\mathcal{N}^p)$ so that for each $\tilde{X} \in T(\mathcal{N}^p)$ we have $F(B\tilde{X}) = B\tilde{X}'$ for some $\tilde{X}' \in T(\mathcal{N}^p)$. If we define a (1,1) tensor field $\tilde{f}$ in $\mathcal{N}^p$ by $\tilde{f}(\tilde{X}) = \tilde{X}'$, then we have

$$F(B\tilde{X}) = B(\tilde{f}\tilde{X}). \tag{2}$$

In the first case we assume that the distribution M is never tangential to $\Phi(\mathcal{N}^p)$ i.e. no vector field of type mX, where m is defined in (2.4) and $X \in T(\Phi(\mathcal{N}^p))$, is tangential to $\Phi(\mathcal{N}^p)$. It shows that any vector field of type mX is independent of any vector field of the form $B\tilde{X}$, $\tilde{X} \in T(\mathcal{N}^p)$.

Applying F to (3.2) we have

$$F^2(B\tilde{X}) = B(\tilde{f}^2\tilde{X}). \tag{3}$$

Now we shall show that vector fields of type $B\tilde{X}$ belong to the distribution L in this case. If we suppose that $m(B\tilde{X}) \neq 0$, then $m(B\tilde{X}) = (I_{T(E)} + F^2)(B\tilde{X}) = B\tilde{X} + F^2(B\tilde{X}) = B\tilde{X} + B(\tilde{f}^2\tilde{X}) = B(\tilde{X} + \tilde{f}^2\tilde{X})$, which, contrary to our assumption, shows that $m(B\tilde{X})$ is tangential to $\Phi(\mathcal{N}^p)$. Therefore $m(B\tilde{X}) = 0$.

Now since $l = -F^2$, from (3.3) and (2.4) we have $B(\tilde{f}^2\tilde{X}) = F^2(B\tilde{X}) = -l(B\tilde{X}) = -(I_{T(E)} - m)(B\tilde{X}) = -B\tilde{X} + m(B\tilde{X})$,
$B(\tilde{f}^2\tilde{X}) = B(-\tilde{X})$, which in view of B being an isomorphism gives $\tilde{f}^2(\tilde{X}) = -\tilde{X}$. Thus we have the following.

Theorem 3.1. An invariant submanifold $\mathcal{N}^p$ imbedded in a $F(3,1)$-structure manifold E^{2n} such that distribution M is never tangential to $\Phi(\mathcal{N}^p)$ is an almost complex manifold.

Theorem 3.2. Let $\mathcal{N}^p$ be an invariant submanifold imbedded in an $F(3,1)$-structure manifold E^{2n} such that the distribution M is never tangential to $\Phi(\mathcal{N}^p)$. If G denotes a metric tensor on E^{2n} given by (2.6), then the induced metric $\tilde{g}$ in $\mathcal{N}^p$ defined by (3.1) is Hermitian.

Proof. $G(X,Y) = G(FX,FY)$ because $F(3,1)$-structure is an isometry on $T(E)$ with respect to G.

$$\tilde{g}(\tilde{f}\tilde{X},\tilde{f}\tilde{Y}) = G(B\tilde{f}\tilde{X},B\tilde{f}\tilde{Y}) = G(FB\tilde{X},FB\tilde{Y}) = \\ = G(B\tilde{X},B\tilde{Y}) = \tilde{g}(\tilde{X},\tilde{Y}).$$

In the second case we assume that the distribution M is always tangential to $\Phi(\mathcal{N}^p)$. It follows, therefore, that $m(B\tilde{X}) = B\tilde{X}^0$, where $\tilde{X} \in T(\mathcal{N}^p)$ for some $\tilde{X}^0 \in T(\mathcal{N}^p)$.

Let us define a (1,1) tensor field $\tilde{m}$ on $T(\mathcal{N}^p)$ such that $\tilde{m}\tilde{X} = \tilde{X}^0$.

We can write $m(B\tilde{X}) = B(\tilde{m}\tilde{X})$. Define a (1,1) tensor field $\tilde{l}$ in $\mathcal{N}^p$ by $\tilde{l} = -\tilde{f}^2$.

Then $B(\tilde{l}\tilde{X}) = B(-\tilde{f}^2\tilde{X}) = -B(\tilde{f}^2\tilde{X}) = -F^2(B\tilde{X}) = l(B\tilde{X})$. Thus we have $B(\tilde{l}\tilde{X}) = l(B\tilde{X})$.

Theorem 3.3. The (1,1) tensor fields $\tilde{l}$ and $\tilde{m}$ on $T(\mathcal{N}^p)$ satisfying

$$\tilde{l} + \tilde{m} = I, \tilde{l}\tilde{m} = \tilde{m}\tilde{l} = 0, \tilde{l}^2 = \tilde{l}, \tilde{m}^2 = \tilde{m}. \tag{4}$$

Proof. We have $l + m = I_{T(E)}$, i.e. $(l+m)(B\tilde{X}) = B\tilde{X}$.

$B(\tilde{l}\tilde{X}) + B(\tilde{m}\tilde{X}) = B\tilde{X}$ or $B(\tilde{l}+\tilde{m})(\tilde{X}) = B\tilde{X}$. Therefore $\tilde{l}+\tilde{m} = \tilde{I}_{T(\mathcal{N}^p)}$. Similary we can prove other relations.

The relation (3.4) show that $\tilde{l}$ and $\tilde{m}$ are complementary projection operators in $\mathcal{N}^p$ given by $\tilde{m} = I_{T(\mathcal{N}^p)} + \tilde{f}^2$, $\tilde{l} = -\tilde{f}^2$. Moreover, we get $B(\tilde{f}^3\tilde{X}) = F^3(B\tilde{X}) = -F(B\tilde{X}) = -B(\tilde{f}\tilde{X})$. Thus we have $\tilde{f}^3 + \tilde{f} = 0$, which shows that in this case $\tilde{f}$ defines an $\tilde{f}(3,1)$-structure on $\mathcal{N}^p$ which we call induced $\tilde{f}(3,1)$-structure. Further, we have $\tilde{g}(\tilde{f}\tilde{X},\tilde{f}\tilde{Y}) = G(B\tilde{f}\tilde{X},B\tilde{f}\tilde{Y}) = G(FB\tilde{X},FB\tilde{Y}) = G(BX,BY) = \tilde{g}(\tilde{X},\tilde{Y})$. Thus we have the following

Theorem 3.4. An invariant submanifold $\mathcal{N}^p$ imbedded in an $F(3,1)$-structure manifold E^{2n} such that the distribution M is always tangential to $\Phi(\mathcal{N}^p)$ is an $\tilde{f}(3,1)$-structure manifold which admits a similar metric $\tilde{g}$ satisfying $\tilde{g}(\tilde{X},\tilde{Y}) = \tilde{g}(\tilde{f}\tilde{X},\tilde{f}\tilde{Y})$.

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