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COMMUTATORS OF C^r–DIFFEOMORPHISMS ACTING ALONG LEAVES

Tomasz Rybicki

Abstract. We study whether a C^r-diffeomorphism which is isotopic to the identity and preserves leaves of a foliation can be written as a product of commutators. The result and techniques depend on the class C^r, where $0 \leq r \leq \infty$.

0. Introduction

Let M be an n-dimensional smooth manifold without boundary, and us denote by $Diff^r(M)_0$ the totality of C^r-diffeomorphisms of M which are isotopic to id through a compactly supported isotopy. Specifically $Diff^r(M)_0$ is the identity component in the C^r topology iff M is compact. A well known theorem of M.R.Herman-W.P.Thurston and J.N.Mather states that $Diff^r(M)_0$ is a simple group, provided $0 \leq r \leq \infty$ and $r \neq n+1$ (cf. [4], [5], [6], [7], [8]). Due to a theorem of D.B.A.Epstein [3] the $Diff^r(M)_0$ is simple if and only if it is perfect, that is equal to its commutator subgroup.

We shall deal with nontransitive groups of diffeomorphisms in the simplest case. Given an (M, F), a k-dimensional smooth foliated structure on M, we let $Diff^r(M, L)_0$ to be the totality of leaf preserving diffeomorphisms isotopic to id through a compactly supported isotopy of leaf preserving diffeomorphisms. The $Diff^r(M, L)_0$ is (usually) nonsimple for obvious reasons, but the question of its perfectness is still valid and important. Our purpose is to extend the mentioned above results and methods of their proofs to the nontransitive case.

Theorem 1. *$Diff^r(M, L)_0$ is perfect provided $0 \leq r \leq k$ or $r = \infty$ (in case $r = 0$ one assumes additionally that $Diff^r(M, L)_0$ possesses a factorization property (1.1)).*

This theorem can be regarded as a starting point to more complicated nontransitive cases, e.g. the group of symplectomorphisms of a Poisson manifold. The

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proof of Theorem 1 is completely different in particular cases. For $r=0$ the proof is a rather simple modification of an argument from [6]. For $r=\infty$ the proof is long and sophisticated. It depends essentially on Implicit Function Theorem in spaces of smooth mappings which enabled Herman [5] to show that $Diff^{\infty}(T^n)_0$ is simple, T^n being the n-dimensional torus , and on the homology theory of Kan simplices by means of which the proof has been extended by Thurston from the torus to an arbitrary M (see e.g. [1]). The proof for $Diff^{\infty}(M,L)_0$ has been presented in [10].

Finally in case $0<r\leq k$ our proof consists in a modification of Mather's argument (cf.[7],[4]). The same argument is valid for the following

Theorem 2. *Let $(\mathbb{R}^n, F_k)$ be the product foliation, i.e. $F_k=\mathbb{R}^k\times\{pt\}$, and let $r\neq n+1$. Then any $f\in Diff^r(\mathbb{R}^n,L)_0$ can be written as a product of commutators $[g,h]=ghg^{-1}h^{-1}$, where g (resp. h) is a leaf (resp. foliation) preserving diffeomorphism of $(\mathbb{R}^n,F_k)$.*

We restrict ourselves to the product foliation for the only reason that there is no possibility of factorization for foliation preserving diffeomorphisms. To our best abilities there is no extension of Theorem 1 to $r>k$ by means of Mather's techniques. Theorem 2 implies only a less significant result.

Namely, let $Diff^r(\mathbb{R}^n,F_k)$ be the group of all foliation preserving diffeomorphisms of $(\mathbb{R}^n,F_k)$. Then any $f\in Diff^r(\mathbb{R}^n,F_k)$ is uniquely decomposed as $f=gh$, where $g\in Diff^r(\mathbb{R}^{n-k})$ and $h\in Diff^r(\mathbb{R}^n,L)$. By $Diff^r(\mathbb{R}^n,F_k)_0$ we denote the totality of f such that $g\in Diff^r(\mathbb{R}^{n-k})_0$, $h\in Diff^r(\mathbb{R}^n,L)_0$.

Corollary. *$Diff^r(\mathbb{R}^n,F_k)_0$ is a perfect group, provided $r\neq n+1, r\neq n-k+1$.*

It is a straightforward consequence of the above decomposition.

Remarks. (1) The problem of the simplicity of $Diff^{n+1}(M)_0$ (cf. [7,III]) is closely related to the perfectness of $Diff^r(M,L)_0$, for $r>k+1$. If one would have that such groups are perfect then $Diff^{n+1}(M)_0$ would be simple. Indeed, let $F\in Diff^{n+1}(\mathbb{R}^n)_0$ be sufficiently near the identity. Then denoting

$$F(x_1,x')=(g(x_1,x'),h(x_1,x')),$$

where $x'=(x_2,\dots,x_n)$, $g:\mathbb{R}^n\to\mathbb{R}, h:\mathbb{R}^n\to\mathbb{R}^{n-1}$, we may assume that $G:(x_1,x')\mapsto(g(x_1,x'),x')$ is a diffeomorphism. Hence if we denote $H(x_1,x')=(x_1,h(G^{-1}(x_1,x'))$ then

$$F(x_1,x')=(g(x_1,x'),h(x_1,x'))=H(g(x_1,x'),x')=H\circ G(x_1,x'),$$

so that $H=F\circ G^{-1}$ is a diffeomorphism. Furthermore, both G and H can be written as a product of commutators, and so can be F.

(2) Theorem 1 still holds for $C^{0,r}$-diffeomorphisms where $r>k$. By a $C^{0,r}$-diffeomorphism we mean a leaf preserving homeomorphism which is of class C^r when restricted to leaves. The proof follows by a further modification of that of Theorem 1.

Throughout we shall assume the acquaintance of some definitions and constructions from [7, I and II] as we shall quote some part of material from this paper.

1. Preliminaries

Due to the following factorization lemma all our considerations can be reduced to the case of the product foliation $(\mathbb{R}^n, F_k)$.

Lemma 1.1. *If $f \in Diff^r(M, L)_0$ is sufficiently near the identity, and $supp(f) \subset U_1 \cup \ldots \cup U_r$, where U_i open balls, then there is a factorization $f = f_s \cdots f_1$ such that $supp(f_j) \subset U_{i(j)}$ for $j = 1, \ldots, s$.*

The proof for $r \geq 1$ is the same as in [9, Lemma 3.1]. In the much more difficult case of homeomorphisms it is not visible how an argument from [2] could be generalized for leaf preserving homeomorphisms.

For U being a finite interval in $\mathbb{R}^n$ we denote

$$Diff^r_U(\mathbb{R}^n, L)_0 = \{f \in Diff^r(\mathbb{R}^n, L)_0 : \exists f_t \text{ with } supp(f_t) \subset U\},$$

where f_t, $t \in I$, is an isotopy in $Diff^r(\mathbb{R}^n, L)$ joining f with id.

For any map $u : \mathbb{R}^n \to \mathbb{R}^n$ of class C^r we have the sequence of seminorms

$$\| u \|_s = sup_x \| D^s u(x) \|, \quad s = 1, \ldots, r,$$

where $D^s u(x)$ denotes the s-th differential of u at x, and $\| \, . \, \|$ is the usual norm on the space of s-linear mappings between normed vector spaces. These seminorms may be infinite in general. However we restrict ourselves to the space of all C^r maps $u : \mathbb{R}^n \to \mathbb{R}^n$ having support contained in a fixed finite interval U. Then $\| \, . \, \|_s$ become norms. Notice that the C^r topology on this space is completely defined by the norm $\| \, . \, \|_r$. Of course $\| u + id \|_s = \| u \|_s$ for $s \geq 2$.

We shall follow Mather [7] but we replace the group $Diff^r(\mathbb{R}^n)_0$ by the leaf preserving one, $Diff^r(\mathbb{R}^n, L)_0$. For simplicity we shall denote

$$Diff(L)_0 = Diff^r(\mathbb{R}^n, L)_0, \quad Diff_U(L)_0 = Diff^r_U(\mathbb{R}^n, L)_0.$$

We begin with some definitions. A *modulus of continuity* is a continuous strictly increasing $\mathbb{R}$-valued function α defined on the interval $[0, \epsilon)$, such that $\alpha(tx) \leq t\alpha(x)$ for any $x \in [0, \epsilon), t \geq 1$ with $tx \in [0, \epsilon)$. Now a mapping $f : (X, \rho) \to (Y, \rho')$ between two metric spaces is called *α-continuous* if there is a constant $C > 0$ such that

$$\rho'(f(x), f(y)) \leq C\alpha(\rho(x, y)).$$

DEFINITION. Let α be a modulus of continuity. A mapping $u : \mathbb{R}^n \to \mathbb{R}^n$ is *of class $C^{r,\alpha}$* if it is of class C^r and its r-th derivative is locally α-continuous.

We have that sums, products, compositions and C^1-inverses of $C^{r,\alpha}$-mappings are still $C^{r,\alpha}$.

We shall denote by $Diff^{r,\alpha}(\mathbb{R}^n)_0$ the identity component of the group of all compactly supported $C^{r,\alpha}$-diffeomorphisms, and $Diff^\alpha(L)_0 = Diff^{r,\alpha}(\mathbb{R}^n)_0 \cap Diff(L)_0, Diff^\alpha_U(L)_0 = Diff^{r,\alpha}(\mathbb{R}^n)_0 \cap Diff_U(L)_0$.

Recall the definition of $\| \,.\, \|_\alpha$, the norm on the space of $C^{r,\alpha}$-mappings

$$\| f \|_\alpha = sup \Big\{ \frac{\| D^r f(x) - D^r f(y) \|}{\alpha(|x-y|)} : x, y \in \mathbb{R}^n, x \neq y \Big\}.$$

The importance of the notion of $C^{r,\alpha}$-smoothness follows on the one hand from the fact that it enables us to apply Schauder Fixed Point Theorem, and on the other that we have

Proposition 1.2. *$Diff^r(\mathbb{R}^n)_0 = \bigcup_\alpha Diff^{r,\alpha}(\mathbb{R}^n)_0$, and, consequently, we get $Diff(L)_0 = \bigcup_\alpha Diff^\alpha(L)_0$, where α runs over all the moduli of continuity.*

2. Proof of Theorems

Following [7,II] we give the proof of Theorem 1 and Theorem 2 for $r \leq n$. The remaining part of Theorem 2 can be shown by making use of [7,I] with rolling-up operators Ψ_i instead of unrolling operators P_i.

From now on we assume that A is a constant greater than 1. For simplicity we shall omit A in the notations. Denote

$$D_i = [-2A, 2A]^{n-i} \times [-2,2]^i \subset \mathbb{R}^n$$

for $i = 1, \dots, n$. Then we have

$$D_n = [-2,2]^n \subset D_{n-1} \subset \cdots \subset D_0 = [-2A, 2A]^n.$$

We shall make use of Mather's operator P_i which "unrolls" diffeomorphisms along the x_i-axis, $i = 1, \dots, n$. Basic features of the definition will be recalled in the next section in order to show the following useful properties of the operators.

(i) The domain of P_i is a C^1-neighbourhood of the id in the group of C^1-diffeomorphisms compactly supported in D_i; the range is in the group of C^1-diffeomorphisms compactly supported in D_{i-1}.

(ii) The P_i preserve the identity.

(iii) If $u \in dom(P_i) \cap Diff^\alpha_{D_i}(L)_0$ then $P_i(u) \in Diff^\alpha_{D_{i-1}}(L)_0$, for $i = 1, \dots, n$.

(iv) $P_i | dom(P_i) \cap Diff^\alpha_{D_i}(L)_0$ is continuous in the C^r topology.

(v) For any $u \in dom(P_i) \cap Diff^\alpha_{D_i}(L)_0$ we have

$$[P_i(u)] = [u] \quad \text{in the abelianization of} \quad Diff^\alpha(L)_0$$

for $i = n-k+1, \dots, n$.

(vi) For any $u \in dom(P_i) \cap Diff^\alpha_{D_i}(L)_0$ we have

$$[P_i(u)] = [u] \quad \text{in the abelianization of} \quad Diff^\alpha(\mathbb{R}^n, F_k)_0$$

for $i = 1, \dots, n-k$.

(vii) There are $\delta > 0$ and independent of A a large constant $C > 0$ such that

$$\| P_i(u)) \|_\alpha \leq CA^{-1} \| u \|_\alpha$$

whenever $u \in dom(P_i) \cap Diff^\alpha(L)_0$ and $\| u \|_\alpha < \delta$ for $i = 1, \dots, n$.

Choose arbitrarily $f \in Diff(L)_0$. Without loss of generality we may assume that $f \in Diff^\alpha_{D_{n-k}}(L)_0$, where α is some modulus of continuity, and $\| f \|_\alpha < \epsilon$, where $\epsilon > 0$ is small. Now we may fix the constants A and ϵ such that

$$3C^k < A^{1+k-r}, \quad 3\epsilon A^{r-1} < \delta,$$

where C, δ are as in (vii).

We set

$$\mathcal{C} = \{u - id \in Diff^\alpha_{D_{n-k}}(L)_0 - id : \| u \|_\alpha \leq \epsilon\}.$$

Then $\mathcal{C}$ is convex and, due to Theorem of Ascoli, compact in the normed space of all leaf preserving C^r-mappings on $\mathbb{R}^n$ compactly supported in D_{n-k}. Let us define a C^r-continuous operator $\Phi : \mathcal{C} \to \mathcal{C}$ in the following way.

Let $\tilde{A}_1 \in Diff^\alpha(L)_0$ be a diffeomorphism which satisfies $\tilde{A}_1(x', x'') = (Ax', x'')$ on D_0, where $x' = (x_1, \dots, x_k), x'' = (x_{k+1}, \dots, x_n)$. Let $\tilde{A}_2 \in Diff^{r,\alpha}(\mathbb{R}^n, F_k)_0$ verify $\tilde{A}_2(x', x'') = (x', Ax'')$ on D_0. We set for $u - id \in \mathcal{C}$

$$u_0 = \tilde{A}_1^{-1} f u \tilde{A}_1$$

and

$$u_1 = P_n(u_0),\ u_2 = P_{n-1}(u_1), \dots, u_k = P_{n-k+1}(u_{k-1}).$$

Define

$$\Phi(u - id) = u_k - id.$$

By (iii) we have $u_k \in Diff^\alpha_{D_{n-k}}(L)_0$. We wish to show $\Phi(\mathcal{C}) \subset \mathcal{C}$.

It follows from the definition of $\| \,.\, \|_\alpha$ that

$$\| u_0 \|_\alpha \leq A^{r-1} \| fu \|_\alpha \leq 3\epsilon A^{r-1}$$

as $\| fu \|_\alpha < 3\epsilon$ (cf. [7]). By making use of (vii) we get

$$\| u_1 \|_\alpha \leq CA^{-1} \| u_0 \|_\alpha,$$
$$\dots$$
$$\| u_k \|_\alpha \leq CA^{-1} \| u_{k-1} \|_\alpha .$$

Therefore

$$\| u_k \|_\alpha \leq C^k A^{-k} \| u_0 \|_\alpha \leq 3\epsilon C^k A^{r-k-1} < \epsilon,$$

as required.

Now Theorem of Schauder states that there exists a fixed "point" of Φ, that is $u_k = u$ for some $u - id \in \mathcal{C}$. Hence the definition of u_0 and (v) imply that $[fu] = [u]$, and consequently $[f] = e$, in the abelianization of $Diff^\alpha(L)_0$. This completes the proof of Theorem 1.

In order to have Theorem 2 for $r \leq n$ one changes some details. The interval D_{n-k} is replaced by D_0, one puts $u_0 = \tilde{A}_1^{-1}\tilde{A}_2^{-1} f u \tilde{A}_2 \tilde{A}_1$, and $u_{i+1} = P_{n-i}(u_i)$, $i = 1, \dots, n$, that is diffeomorphisms are unrolled n times. Finally we use both (v) and (vi).

3. Properties of the operators P_i

First we give an outline of the construction of P_i. We write C_i for $\mathbb{R}^{i-1}\times S^1\times\mathbb{R}^{n-i}$, i.e. C_i is the set of n-tuples $(x_1,\dots,\vartheta_i,\dots,x_n)$ where ϑ_i is a real defined mod 1. Let $\pi:\mathbb{R}^n\to C_i$ be the canonical projection. Notice that there is the obvious action of S^1 on C_i, and denote by $\mathcal{G}_i$ the group of equivariant diffeomorphisms with respect to this action.

Let $u\in Diff^{r,\alpha}_{D_i}(n+1)_0$ and $\| u-id\|_1<\frac{1}{2}$. We define a diffeomorphism $\Gamma_i(u): C_i\to C_i$ in the following way. For $\vartheta\in C_i$ we choose $x\in\mathbb{R}^n$ such that $\pi(x)=\vartheta$ and $x_i<-2$. Next we choose an integer $N>0$ so large that

$$(T_i u)^N(x)_i>2.$$

Here $T_i:\mathbb{R}^n\to\mathbb{R}^n$ denotes the one time transformation of the vector field $\partial_i=\partial/\partial x_i$, that is the unit translation along the x_i-axis. It is visible that $N=10$ is sufficient. Then one defines

$$\Gamma_i(u)(\vartheta) \;=\; \pi(T_i u)^N(x).$$

In particular, if $|x_j|>2A$ for some $j\neq i$ then $\Gamma_i(u)(\vartheta)=\vartheta$. One can also check that the operation $u\mapsto\Gamma_i(u)$ is continuous.

For $\Gamma_i(u)$ sufficiently near id there is a unique S^1-equivariant diffeomorphism $g:C_i\to C_i$ such that

$$g\mid\{\vartheta_i=0\} \;=\; \Gamma_i(u)\mid\{\vartheta_i=0\}.$$

Let $h=g^{-1}\Gamma_i(u)$. Next let $\tilde h$ be the unique diffeomorphism of $\mathbb{R}^n$ verifying $\pi\tilde h=h\pi$ and $\tilde h=id$ on $\{x_i=0\}$.

Denoting by B a sufficiently large positive integer, and assuming that $\tilde h$ (and u) is C^1-close to id one defines diffeomorphisms $\tilde h_1,\dots,\tilde h_B$ of $\mathbb{R}^n$ as follows

$$\begin{aligned}
\tilde h_1-id&=\frac{1}{B}(\tilde h-id),\\
\tilde h_2-id&=\frac{1}{B-1}(\tilde h\tilde h_1^{-1}-id),\\
&\dots\\
\tilde h_{B-1}-id&=\frac{1}{2}(\tilde h\tilde h_1^{-1}\dots\tilde h_{B-2}^{-1}-id),\\
\tilde h_B-id&=\tilde h\tilde h_1^{-1}\dots\tilde h_{B-1}^{-1}-id,
\end{aligned}$$

so that $\tilde h=\tilde h_B\tilde h_{B-1}\dots\tilde h_1$. (This decomposition enables us to get (vii).)

Now one chooses a function $\zeta\in C^\infty(\mathbb{R})$ which takes values in [0,1], is periodic of period 1, equal to 1 in a neighbourhood of the integers, and equal to 0 in a neighbourhood of the half-integers. We define $\tilde h_{jk}$ where $j=1,\dots,B$, $k=0,1$, by the equalities

$$\tilde h_{j0}-id=\zeta(x_i)(\tilde h_j-id)\quad\text{and}\quad\tilde h_{j1}=\tilde h_j\tilde h_{j0}^{-1}.$$

Thus we get a decomposition $\tilde{h} = \tilde{h}_{B1}\tilde{h}_{B0}\dots\tilde{h}_{1,1}\tilde{h}_{1,0}$.

Next one defines a sequence of strips "regularly" located in the wide strip $|x_i| < 2A$ as follows. Letting a be the least half-integer $\geq -2A$ one denotes

$$E^j_- = \{x \in \mathbb{R}^n : a + 3j - 3 \leq x_i \leq a + 3j - 2\},$$
$$E^j_+ = \{x \in \mathbb{R}^n : a + 3j - 3/2 \leq x_i \leq a + 3j - 1/2\}.$$

Then B is defined as the greatest integer such that $a + 3B - 1/2 \leq 2A$.

Finally one defines $v = P_i(u)$ as a unique diffeomorphism of $\mathbb{R}^n$ satisfying

$$v|E^j_- = \tilde{h}_{j0}|E^j_-, \quad v|E^j_+ = \tilde{h}_{j1}|E^j_+$$

and

$$v|\big(\mathbb{R}^n - \bigcup_j E^j_- \cup E^j_+\big) \;=\; id.$$

This terminates the construction of P_i.

Remark. The clue property of Mather's construction is that

$$\Gamma_i(P_i(u))\Gamma_i(u)^{-1} \in \mathcal{G}_i.$$

This fact enables to obtain suitable conjugate relations (Proposition 3.2).

Proposition 3.1. *If $u \in dom(P_i)$, $i = 1,\dots,n$, and u preserves the leaves of $\mathcal{F}$ then so does $P_i(u)$.*

Proof consists in looking closely at the definition of P_i. In both cases: $i \leq k$, and $i > k$, successive steps are "leaf preserving" (one considers also the induced foliation on C_i). In case $i \leq k$ the $(T_iu)^N$ is leaf preserving as T_i and u are so, and so is $\Gamma_i(u)$. In case $i > k$ the $\Gamma_i(u)$ takes the form $\Gamma_i(u)(x',\theta') = (u_1(x'),\theta')$ for any $\theta = (x',\theta') \in C_n$, where $\theta' = (x_{k+1},\dots,\theta_i,\dots,x_n)$. This follows from the fact that T_i is the unit translation and θ' is mod 1. Hence $\Gamma_i(u)$ is leaf preserving. It is even more clear that the remaining steps are also "leaf preserving" (cf.[7,II]).

Choose a function $\rho \in C^\infty(\mathbb{R})$ such that $\rho = 1$ on $[-2A, 2A]$ and $supp(\rho) \subset [-2A-1, 2A+1]$. Next, abusing the notation, we define $\rho \in C^\infty(\mathbb{R}^n)$ by $\rho(x) = \rho(x_1)\dots\rho(x_n)$, and $\rho \in C^\infty(\mathbb{R}^{n-k})$ by $\rho(x'') = \rho(x_{k+1})\dots\rho(x_n)$ where $x'' = (x_{k+1},\dots,x_n)$. We set

$$\tau_i(x) = exp(\rho(x)\partial_i), \quad i = 1,\dots,k,$$
$$\tau_i(x) = exp(\rho(x'')\partial_i), \quad i = k+1,\dots,n.$$

Then $\tau_i \in Diff^\alpha(L)_0$, $i \leq k$ and $\tau_i \in Diff^{r,\alpha}(\mathbb{R}^n, F_k)_0$, $i > k$.

Next let ϕ_i, $i = 1,\dots,n$, be diffeomorphisms characterized by the following properties: (a) $dom(\phi_i) = \{x \in \mathbb{R}^n : |x_j| \leq 2A, i \neq j\}$, (b) $\phi_{i*}(\partial_i) = \rho\partial_i$, (c) $\phi|D_0 = id$, (d) in case $i \leq k$ the ϕ_i is leaf preserving, (e) in case $i > k$ the ϕ_i is foliation preserving.

Note that

$$\tau_i = \phi_i T_i \phi_i^{-1}, \quad i = 1, \dots, n, \tag{1}$$

since $T_i = exp(\partial_i), \tau_i = exp(\rho\partial_i)$.

The following is a modification of [7,&4], and we reproduce only a sketch of the proof.

Proposition 3.2. *Let $i = 1, \dots, n$ and let $u, v \in Diff^{\alpha}_{D_0}(L)_0$. If u and v are sufficiently C^1-close to the identity and $\Gamma_i(v)\Gamma_i(u)^{-1} \in \mathcal{G}_i$ then $\lambda\tau_i u\lambda^{-1} = \tau_i v$ where $\lambda \in Diff^{\alpha}(L)_0$.*

Proof. Following [7] we put

$$\Lambda(x) = (T_i v)^N (T_i u)^{-N}(x),$$

where N is a positive integer so large that $(T_i u)^{-N}(x)_i < 2A$. For any $x \in \mathbb{R}^n$ such an N exists, and the definition is independent of N (large). Observe that $supp\Lambda \subset \{x \in \mathbb{R}^n : x_i > -2A, |x_j| < 2A, j \neq i\}$. It follows from the definition

$$\Lambda T_i u \Lambda^{-1} = T_i v \tag{2}$$

and

$$\Gamma_i(v)\Gamma_i(u)^{-1}\pi = \pi\Lambda. \tag{3}$$

Furthermore Λ is a leaf preserving diffeomorphism, that is

$$\Lambda(x)_i = x_i \, , i > k. \tag{4}$$

In fact, in case $i \leq k$, T_i, u, v are leaf preserving, and so is Λ. In case $i > k$ the factor $(T_i u)^{-N}$ sends the leaf $x_j = a_j$ to the leaf $x_j = a_j - N_j$, where $N_j = N$ for $j = i$ and $N_j = 0$ for $j \neq i$, which is sent back by $(T_i v)^N$ to $x_j = a_j$. It is then clear that Λ changes nothing in the transversal directions.

The assumption $\Gamma_i(v)\Gamma_i(u)^{-1} \in \mathcal{G}_i$ and (3) yield the existence of $\Lambda_j : \mathbb{R}^n \to \mathbb{R}$, $j \neq i$, satisfying

$$\Lambda_j(x_i') = \Lambda(x)_j$$

for $x_i > 2A$, where $x_i' = (x_1, \dots, \hat{x}_i, \dots, x_n)$. Next $\Lambda'(x_i') = (\Lambda_1(x_i'), \dots, \Lambda_n(x_i'))$, and let Λ_t' be a leaf preserving isotopy from Λ' to id supported in the set $\{|x_j| < 2A, j \neq i\}$.

Now the definition of λ is the following (in the second and third line the i-th coordinate is written last)

$$\begin{aligned}
\lambda(x) &= \phi_i \Lambda \phi_i^{-1}, \quad x \in im(\phi_i) \\
&= (\Lambda'(x_i'), exp(\tilde{\rho}(\Lambda(x)_i - x_i)\partial_i)(x)_i), \quad 2A \leq x_i \leq 2A + 2 \\
&= (\Lambda_t'(x_i'), x_i), \quad 2A + 2 \leq x_i \leq 2A + 3 \\
&= x \quad \text{otherwise.}
\end{aligned}$$

It is easy to see that λ is leaf preserving. Indeed, the r.h.s. of the first line is so as Λ and $\phi_i, i \leq k$, is leaf preserving, and $\phi_i, , i > k$ is foliation preserving. The r.h.s. of the second line is leaf preserving because of (4). Finally the r.h.s. of the third line is leaf preserving because of the assumption on Λ'_t.

The proof is concluded by the observation that conjugating by ϕ_i the equality (2) and making use of (1) imply the desired equality $\lambda\tau_i u\lambda^{-1} = \tau_i v$.

Now it is visible that the properties (i-vii) hold. In fact, (i),(ii),(iv) follows by definition. Proposition 3.1 coincides with (iii), and Proposition 3.2 implies (v),(vi) as $\tau_i \in Diff^{\alpha}(L)_0$ for $i \leq k$, and $\tau_i \in Diff^{r,\alpha}(\mathbb{R}^n, F_k)_0$ for $i > k$. For the proof of (vii) see [7,&4].

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Institute of Mathematics, Pedagogical University
ul.Rejtana 16 A, 35-310 Rzeszów, POLAND

E-mail: rybicki@im.uj.edu.pl