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SMOOTH CONNECTIONS AND HORIZONTAL DISTRIBUTIONS ON MANIFOLDS OVER LOCAL ALGEBRAS

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Abstract. On manifolds over local algebras special distributions complementary to leaves of the canonical foliation and smooth connections are considered. The main purpose is to study obstructions to existence of such distributions in terms of the Atiyah classes of foliated and smooth principal bundles. Besides that the notion of holonomy of a manifold over local algebra is introduced, and it is shown that existence of complementary distributions under consideration on a manifold over local algebra depends on its holonomy.

§1. Introduction

As it was shown by M.Atiyah [1], in order that complex analytic principal bundle admit a complex analytic connection it is necessary and sufficient that a certain cohomology class, called the Atiyah class, vanish. The present paper is devoted to the problem of existence of smooth connections and special horizontal distributions, which become connections in the case of A.Weil's bundles of infinitely near points [2] (see also [3],[4]), on manifolds over local algebras. There are canonical foliations on manifolds over local algebras, and obstructions to existence of smooth connections and special horizontal distributions are related with obstructions to existence of projectable connections in foliated principle bundles defined by P.Molino [5] and by F.W.Kamber and P.Tondeur [6].

We adopt the following notation (see [7]):

$\mathbb{R}(N,q)$ is the algebra of truncated real polynomials of degree q in N variables,

$\mathbb{A} = \mathbb{R}(N,q)/\mathbb{I}$ a local algebra of width N and height q in the sense of A.Weil (see [2], [3], [4]),

$\overset{\circ}{\mathbb{A}}$ the maximal ideal of $\mathbb{A}$,

$\mathbb{A}^n$ the $\mathbb{A}$-module of n-tuples of elements of $\mathbb{A}$,

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$\overset{\circ}{\mathbb{A}}{}^n$ the submodule of $\mathbb{A}^n$ consisting of n-tuples of elements of $\overset{\circ}{\mathbb{A}}$,

$\pi_0^q : \mathbb{A} \to \mathbb{R}$ and $\pi_0^q : \mathbb{A}^n \to \mathbb{R}^n$ the natural epimorphisms (for simplicity we use the same symbol π_0^q).

The module $\mathbb{A}^n$ (and the algebra $\mathbb{A}$, in particular) can be represented as the direct sum of vector spaces $\mathbb{A}^n = \mathbb{R}^n \oplus \overset{\circ}{\mathbb{A}}{}^n$, and so every $X \in \mathbb{A}^n$ can be uniquely represented in the form $X = x + \overset{\circ}{X}$ where $x \in \mathbb{R}^n$ and $\overset{\circ}{X} \in \overset{\circ}{\mathbb{A}}{}^n$. The submersion $\pi_0^q : \mathbb{A}^n \to \mathbb{R}^n$ defines a foliation on $\mathbb{A}^n$ whose leaves are cosets of $\overset{\circ}{\mathbb{A}}{}^n$. We shall call it *the* $\overset{\circ}{\mathbb{A}}$*-foliation*.

A smooth mapping

$$\Phi : U \subset \mathbb{A}^n \to U' \subset \mathbb{A}^k \tag{1}$$

is called $\mathbb{A}$-*smooth* if the tangent mapping $T_X \Phi$ is $\mathbb{A}$-linear for any $X \in U$. If U is a simple open subset [5] for the $\overset{\circ}{\mathbb{A}}$-foliation, an $\mathbb{A}$-smooth mapping is of the form

$$Y^{i'} = \phi^{i'}(x^i) + \sum_{|p|=1}^{q} \frac{1}{p!} \frac{D^p \phi^{i'}}{Dx^p} \overset{\circ}{X}{}^p \tag{2}$$

where $i = 1, \ldots, n$, $i' = 1, \ldots, k$, $p = (p_1 \ldots, p_n)$ is a multiindex of length n, and $\phi^{i'} : \pi_0^q(U) \to \mathbb{A}$ are $\mathbb{A}$-valued smooth functions. In this case the mapping (1) is uniquely extended to the $\mathbb{A}$-smooth mapping

$$\Phi : (\pi_0^q)^{-1}(\pi_0^q(U)) \to (\pi_0^q)^{-1}(\pi_0^q(U')) \tag{3}$$

given by the same equations (2). Moreover, if the mapping (1) is an $\mathbb{A}$-diffeomorphism (for $n = k$), the mapping (3) is an $\mathbb{A}$-diffeomorphism too. If the domain U of a mapping (1) is not simple, it can be given by the formula (2) where $\phi^{i'} : U \to \mathbb{A}$ are basic (projectable) functions for $\overset{\circ}{\mathbb{A}}$-foliation.

Every $\mathbb{A}$-diffeomorphism Φ of the form (3) maps leaves of the $\overset{\circ}{\mathbb{A}}$-foliation into leaves. In particular, if $\pi_0^q(\Phi(0)) = 0$, the restriction $\overset{\circ}{\Phi} = \Phi|\overset{\circ}{\mathbb{A}}{}^n : \overset{\circ}{\mathbb{A}}{}^n \to \overset{\circ}{\mathbb{A}}{}^n$ is of the form

$$\overset{\circ}{Y}{}^i = \sum_{|p|=0}^{q} \phi_p^i \overset{\circ}{X}{}^p \tag{4}$$

where $\phi_0^i \in \overset{\circ}{\mathbb{A}}$, $\phi_p^i \in \mathbb{A}$.

Let $D_n(\mathbb{A})$ denote the set of all diffeomorphisms of the form (4). $D_n(\mathbb{A})$ is a Lie group, the relations $\phi_0^i = 0$, $\phi_p^i \in \mathbb{R}$ determine a Lie subgroup in $D_n(\mathbb{A})$ which is isomorphic to the differential group G_n^q (see [4]). $D_n(\mathbb{A})$ is the semidirect product $D_n(\mathbb{A}) = G_n^q \times \overset{\circ}{D}_n(\mathbb{A})$ where the invariant subgroup $\overset{\circ}{D}_n(\mathbb{A})$ is defined by the relations $\pi_0^q(\phi_p^i) = 0$ for $|p| > 1$, $\pi_0^q(\phi_j^i) = \delta_j^i$ [8]. We denote by $\tau : D_n(\mathbb{A}) \to G_n^q$ the natural epimorphism, and by $\mathfrak{d}_n(\mathbb{A})$ the Lie algebra of $D_n(\mathbb{A})$.

By an n-dimensional *smooth manifold over* $\mathbb{A}$ *($\mathbb{A}$-smooth manifold)* we shall mean (see [9], [7]) a smooth manifold $M^{\mathbb{A}}$ with a maximal atlas of charts $h_\alpha : U_\alpha \subset M^{\mathbb{A}} \to U'_\alpha \subset \mathbb{A}^n$ with $\mathbb{A}$-smooth coordinate changes $h_\alpha \circ h_\beta^{-1}$. Such charts are called $\mathbb{A}$*-charts* and an atlas is called *an* $\mathbb{A}$*-atlas*. The bundle $J^{\mathbb{A}}M$ of infinitely near points of $\mathbb{A}$-type in the sense of A.Weil [2] ($\mathbb{A}$-jets [4], [7]) is an example of n-dimensional $\mathbb{A}$-smooth manifold [3].

Let

$$X^{i'} = \Phi^{i'}(X^i) \tag{5}$$

be an $\mathbb{A}$-smooth coordinate change on an $\mathbb{A}$-smooth manifold $M^{\mathbb{A}}$. From (2) it follows that the real parts $x^{i'}$ of the coordinates $X^{i'}$ locally depend only on the real parts x^i of the coordinates X^i, and so the equations $x^i = const$ determine a foliation on $M^{\mathbb{A}}$. This foliation is called *the canonical* $\mathring{\mathbb{A}}$*-foliation*. Note that each submodule of $\mathbb{A}^n$ invariant under $\mathbb{A}$-linear transformations generates a foliation on $M^{\mathbb{A}}$ [7], and so $M^{\mathbb{A}}$ is a multifoliated manifold.

When we fix x^i in (5), we obtain coordinate transformations on leaves of the canonical $\mathring{\mathbb{A}}$-foliation. Locally these transformations are of the form (4), therefore each leaf of $M^{\mathbb{A}}$ carries a structure of (X, G)-manifold in the sense of Thurston (see [10]) for $X = \mathring{\mathbb{A}}^n$ and $G = D_n(\mathbb{A})$, and $M^{\mathbb{A}}$ carries a structure of tangential $(\mathring{\mathbb{A}}^n, D_n(\mathbb{A}))$-foliation (see [11], [12] about tangential (X, G)-foliations).

§2. Holonomy of a manifold over local algebra

Consider now an $\mathbb{A}$-smooth manifold with an $\mathbb{A}$-atlas $\{h_\alpha : U_\alpha \subset M^{\mathbb{A}} \to U'_\alpha \subset \mathbb{A}^n\}_{\alpha\in A}$. One can define the following two fiber bundles over $M^{\mathbb{A}}$ with standard fiber $\mathring{\mathbb{A}}^n$ and structure groups $D_n(\mathbb{A})$ and G_n^q respectively:

1) $\pi : O^V M^{\mathbb{A}} \to M^{\mathbb{A}}$ with the atlas $\{H_\alpha : \pi^{-1}(U_\alpha) \to U'_\alpha \times \mathring{\mathbb{A}}^n\}_{\alpha\in A}$ whose transition functions $H_\alpha \circ H_\beta^{-1}$ are of the form

$$\{X^i_\beta, \overset{*}{X}{}^i_\beta\} \mapsto \{X^i_\alpha = \phi^i_{\alpha\beta} + \sum_{|p|=1}^{q} \frac{1}{p!}\frac{D^p\phi^i_{\alpha\beta}}{Dx^p_\beta}\mathring{X}^p_\beta,\ \overset{*}{X}{}^i_\alpha = \mathring{\phi}^i_{\alpha\beta} + \sum_{|p|=1}^{q} \frac{1}{p!}\frac{D^p\phi^i_{\alpha\beta}}{Dx^p_\beta}\overset{*}{X}{}^p_\beta\}.$$

2) $\tilde\pi : \tilde O^V M^{\mathbb{A}} \to M^{\mathbb{A}}$ with the atlas $\{\tilde H_\alpha : \tilde\pi^{-1}(U_\alpha) \to U'_\alpha \times \mathring{\mathbb{A}}^n\}_{\alpha\in A}$ whose transition functions $\tilde H_\alpha \circ \tilde H_\beta^{-1}$ are of the form

$$\{X^i_\beta, \hat X^i_\beta\} \mapsto \{X^i_\alpha = \phi^i_{\alpha\beta} + \sum_{|p|=1}^{q} \frac{1}{p!}\frac{D^p\phi^i_{\alpha\beta}}{Dx^p_\beta}\mathring{X}^p_\beta,\ \hat X^i_\alpha = \sum_{|p|=1}^{q} \frac{1}{p!}\frac{D^p\bar\phi^i_{\alpha\beta}}{Dx^p_\beta}\hat X^p_\beta\}$$

where $\phi_{\alpha\beta} = h_\alpha \circ h_\beta^{-1}$, $\bar\phi_{\alpha\beta} = \pi^q_0 \circ \phi_{\alpha\beta}$. The charts H_α and $\tilde H_\alpha$ will be called *the charts corresponding to* h_α.

The bundle $\tilde O^V M^{\mathbb{A}}$ can be identified with the bundle $J^{\mathbb{A}}_{tr} M^{\mathbb{A}}$ of transversal $\mathbb{A}$-jets over $M^{\mathbb{A}}$ with respect to the canonical $\mathring{\mathbb{A}}$-foliation on $M^{\mathbb{A}}$. In the case when

$M^{\mathbb{A}}$ is *complete*, that is all leaves of the canonical $\mathring{\mathbb{A}}$-foliation are complete, the bundle $O^V M^{\mathbb{A}}$ can be realized as the union of fundamental groupoids of leaves (see [12]). By $P^V M^{\mathbb{A}}$ we shall denote the principal $D_n(\mathbb{A})$-bundle associated with $O^V M^{\mathbb{A}}$. $P^V M^{\mathbb{A}}$ can be realized as the set of germs of local $\mathbb{A}$-affine mappings from $\mathring{\mathbb{A}}^n$ into leaves of the canonical $\mathring{\mathbb{A}}$-foliation. The bundle $J_{tr}M^{\mathbb{A}}$ is associated with the bundle $B^q_{tr}M^{\mathbb{A}}$ of transversal q-frames.

Let L be a leaf of the canonical $\mathring{\mathbb{A}}$-foliation through $X_0 \in M^{\mathbb{A}}$, $\gamma : [0,1] \to L$ a continuous loop based at X_0, and (U, h) a fixed $\mathbb{A}$-chart on $M^{\mathbb{A}}$ with $h(X_0) = 0$ and U being simple open subset. Consider a collection of $\mathbb{A}$-charts $\{U_{\alpha_k}, h_{\alpha_k}\}$, $k = 1, \dots, s$, on $M^{\mathbb{A}}$ satisfying the following conditions: U_{α_k} is simple open set, $U_{\alpha_1} = U_{\alpha_s} = U$, $h_{\alpha_k}(U_{\alpha_k} \cap L) \subset \mathring{\mathbb{A}}^n$, and there is a subdivision $t_0 = 0 < t_1 < \dots < t_s = 1$ of $[0,1]$ such that $\gamma([t_k, t_{k+1}]) \subset U_{k+1}$. Each $\mathbb{A}$-smooth coordinate change $\Phi_k = h_{\alpha_k} \circ h^{-1}_{\alpha_{k+1}}$ restricted on a simple neiborhood of the point $\gamma(t_k)$ can be extended to an $\mathbb{A}$-diffeomorphism Φ_k of the form (3), therefore on a suitable domain $(\pi^q_0)^{-1}(V)$, where V is an open neighborhood of $0 \in \mathbb{R}^n$, we can take the composition $\Phi_0 \circ \Phi_1 \circ \dots \circ \Phi_s$. The germ $\alpha(\gamma)$ of this composition is defined along the entire submodule $\mathring{\mathbb{A}}^n$ and depends only on a homotopy class of γ, and if $\gamma_3 = \gamma_2 * \gamma_1$, then $\alpha(\gamma_3) = \alpha(\gamma_2) \circ \alpha(\gamma_1)$. Thus we obtain a homomorphism

$$\alpha : \pi_1(L, X_0) \ni [\gamma] \mapsto \alpha(\gamma) \in \mathrm{Diff}_{\mathbb{A}}(\mathbb{A}^n, \mathring{\mathbb{A}}^n)$$

where $\mathrm{Diff}_{\mathbb{A}}(\mathbb{A}^n, \mathring{\mathbb{A}}^n)$ is the group of germs of $\mathbb{A}$-smooth diffeomorphisms of the form $(\pi^q_0)^{-1}(V) \subset \mathbb{A}^n \to (\pi^q_0)^{-1}(V') \subset \mathbb{A}^n$, and V and V' are neighborhoods of $0 \in \mathbb{R}^n$.

We shall call α *the holonomy representation of* $M^{\mathbb{A}}$ *at* X_0. The image of α will be called *the holonomy group of* $M^{\mathbb{A}}$ *at* X_0.

Besides α the following three holonomy representations are associated with a leaf L of the canonical $\mathring{\mathbb{A}}$-foliation: α_{tr}, the holonomy representation of L at X_0 in the sense of the theory of foliations [5], α_l, the holonomy of L as an $(\mathring{\mathbb{A}}^n, D_n(\mathbb{A}))$-manifold ([10], §2.2), and $\bar{\alpha}_l = \tau \circ \alpha_l$, which will be called *the* G^q_n*-holonomy representation* of L.

Theorem 1. *The holonomy representations associated with* $M^{\mathbb{A}}$ *are related by the equalities*

$$\alpha_{tr} = t \circ \alpha, \ \alpha_l = r \circ \alpha, \ \bar{\alpha}_l = j^q \circ \alpha_{tr} \tag{6}$$

where t *maps a germ* Φ *of* $\mathbb{A}$*-smooth diffeomorphism into the germ* $\pi^q_0 \circ \Phi|(\mathbb{R}^n, 0)$, r *is the restriction of germs to the leaf* $\mathring{\mathbb{A}}^n \subset \mathbb{A}^n$ *,and* j^q *assigns to a germ its* q*-jet.*

Proof. The relations (6) follow from the structure of equations (5), which provides the commutativity of the following diagram

$$\begin{tikzcd}
\mathrm{Diff}_{\mathbb{A}}(\mathbb{A}^n, \overset{\circ}{\mathbb{A}}{}^n) \arrow[r, "t"] \arrow[d, "r"'] & \mathrm{Diff}(\mathbb{R}^n, 0) \arrow[d, "j^q"] \\
D_n(\mathbb{A}) \arrow[r, "\tau"] & G_n^q
\end{tikzcd}$$

Note. In the case of algebra $\mathbb{R}(\varepsilon)$ a leaf L of the canonical $\overset{\circ}{\mathbb{R}}(\varepsilon)$-foliation carries a structure of affine manifold and $\bar{\alpha}_l$ is the linear holonomy representation [13] of L as an affine manifold. Relations similar to $\bar{\alpha}_l = j^q \circ \alpha_{tr}$ have been found by T.Inaba [11] for Lagrangian and Legendrian foliations.

The holonomy representation α of $M^{\mathbb{A}}$ at X_0 depends on a choice of fixed $\mathbb{A}$-chart (U, h). An interpretation of this holonomy independent of such a choice can be given in terms of the bundles $O^V M$ and $P^V M$. Let us assign to a germ $\Phi \in \mathrm{Diff}_{\mathbb{A}}(\mathbb{A}^n, \overset{\circ}{\mathbb{A}}{}^n)$ written in coordinates in the form

$$Y^i = \phi^i(x^j) + \sum_{|p|=1}^{q} \frac{1}{p!} \frac{D^p \phi^i}{Dx^p} \overset{\circ}{X}{}^p$$

the germ of projectable diffeomorphism $l(\Phi) : (O^V \mathbb{A}^n, \pi^{-1}(0)) \to (O^V \mathbb{A}^n, \pi^{-1}(0))$ given by the relations

$$\{x^i, \overset{\circ}{X}{}^i, \overset{*}{X}{}^i\} \mapsto \{y^i = \bar{\phi}^i(x^j), \overset{\circ}{Y}{}^i = \overset{\circ}{X}{}^i, \overset{*}{Y}{}^i = \overset{\circ}{\phi}{}^i(x^j) + \sum_{|p|=1}^{q} \frac{1}{p!} \frac{D^p \phi^i}{Dx^p} \overset{*}{X}{}^p\}. \tag{7}$$

Let H be the chart corresponding to the fixed chart h which was used in the definition of holonomy representation, and $\mathrm{Diff}_{\mathbb{A},pr}(O^V M^{\mathbb{A}}, \pi^{-1}(X_0))$ the group of germs of projectable bundle diffeomorphisms which have the form (7) with respect to charts of the atlas $\{H_\alpha, \pi^{-1}(U_\alpha)\}$. We define the holonomy representation

$$\alpha_O : \pi_1(L, X_0) \to \mathrm{Diff}_{\mathbb{A},pr}(O^V M^{\mathbb{A}}, \pi^{-1}(X_0))$$

as follows: $\alpha_O([\gamma]) = H^{-1} l(\alpha(\gamma)) H$. Passing to the principle bundle $P^V M^{\mathbb{A}}$ we define in the same way the holonomy representation

$$\alpha_P : \pi_1(L, X_0) \to \mathrm{Diff}_{\mathbb{A},pr}(P^V M^{\mathbb{A}}, \pi^{-1}(X_0)).$$

§3. $\mathbb{A}$-affine horizontal distributions on $M^{\mathbb{A}}$

The bundle $J^{\mathbb{A}} M$ can be treated as a bundle with structure group $D_n(\mathbb{A})$ [8]. A connection ∇ in the corresponding principal $D_n(\mathbb{A})$-bundle is called *an $\mathbb{A}$-affine connection.* The horizontal distribution on $J^{\mathbb{A}} M$ induced by the connection ∇ is given in local $\mathbb{A}$-coordinates $\{X^i\}$ on $J^{\mathbb{A}} M$ by the equations

$$d\overset{\circ}{X}{}^i - \sum_{|p|=0}^{q} \Gamma^i_{j\,p} \overset{\circ}{X}{}^p \, dx^j = 0 \tag{8}$$

where $\Gamma^i_{j\,p} = \Gamma^i_{j\,p}(x^k)$ are $\mathbb{A}/(\overset{\circ}{\mathbb{A}}{}^n)^{q-|p|+1}$-valued connection coefficients [8]. The transformation law of the coefficients $\Gamma^i_{j\,p}$ is derived in [14].

Definition[14]**.** A distribution ∇ on an $\mathbb{A}$-smooth manifold $M^{\mathbb{A}}$ complementary to leaves of the canonical $\overset{\circ}{\mathbb{A}}$-foliation is called *an $\mathbb{A}$-affine horizontal distribution* if in terms of $\mathbb{A}$-charts (U_α, h_α), $\alpha \in A$, it is given by equations of the form (8), where $\Gamma^i_{j\,p}$ are projectable functions with respect to the canonical $\overset{\circ}{\mathbb{A}}$-foliation.

An $\mathbb{A}$-affine horizontal distribution (8) determines a horizontal distribution on the bundle $O^V M^{\mathbb{A}}$ given by the equations

$$d\overset{*}{X}{}^i - \sum_{|p|=0}^{q} \Gamma^i_{j\,p} \overset{*}{X}{}^p \, dx^j = 0 \tag{9}$$

in terms of charts $(\pi^{-1}(U_\alpha), H_\alpha)$, $\alpha \in A$. The distribution (9) is associated with a projectable [5] (with respect to the canonical $\overset{\circ}{\mathbb{A}}$-foliation) connection ∇^V in the foliated principal bundle $P^V M^{\mathbb{A}}$. Conversely, a projectable connection ∇^V in the principal bundle $P^V M^{\mathbb{A}}$ determines uniquely an $\mathbb{A}$-affine horizontal distribution, consequently $M^{\mathbb{A}}$ admits an $\mathbb{A}$-affine horizontal distribution if and only if there is a projectable connection in $P^V M^{\mathbb{A}}$. A necessary and sufficient condition for existence of such a connection is vanishing of the Atiyah class [5] $a(P^V M^{\mathbb{A}})$ in the d_F-cohomology group $H^{1,1}_{ad\,F}(P^V M^{\mathbb{A}}, \mathfrak{d}_n(\mathbb{A}))$ of tensorial forms of ad-type on $P^V M^{\mathbb{A}}$ where ad is the adjoint representation of the group $D_n(\mathbb{A})$ in its Lie algebra $\mathfrak{d}_n(\mathbb{A})$.

The difference $\omega_1 - \omega_2$ of connection forms of two projectable connections ∇^V_1 and ∇^V_2 is a tensorial form of type $(0,1)$, which can be treated as a d_F-cohomology class belonging to $H^{0,1}_{ad\,F}(P^V M^{\mathbb{A}}, \mathfrak{d}_n(\mathbb{A}))$. Therefore, in case when $M^{\mathbb{A}}$ admits an $\mathbb{A}$-affine horizontal distribution, the set of all such distributions is in one-to-one correspondence with the group $H^{0,1}_{ad\,F}(P^V M^{\mathbb{A}}, \mathfrak{d}_n(\mathbb{A}))$.

The holonomy of $M^{\mathbb{A}}$ may be an obstruction to existence of an $\mathbb{A}$-affine horizontal distribution. Let Z_0 be an element of a fiber $P^V_{X_0} M^{\mathbb{A}}$ over $X_0 \in M^{\mathbb{A}}$, $Q_{Z_0} \subset T_{Z_0} P^V M^{\mathbb{A}}$ the horizontal subspace of projectable connection ∇^V, and $[\gamma] \in \pi_1(L, X_0)$. Then the tangent mapping $T_{Z_0}\alpha_P(\gamma)$ maps Q_{Z_0} into $Q_{R_g(Z_0)}$ where $g \in D_n(\mathbb{A})$ is the element determined by the relation $\alpha_P(\gamma)(Z_0) = R_g(Z_0)$, hence Q_{Z_0} is invariant under the mapping $T_{Z_0}\alpha_P(\gamma) \circ R_{g^{-1}}$. All such mappings constitute a subgroup $\alpha_{P,Z_0} \subset GL(T_{Z_0} P^V M^{\mathbb{A}})$.

Summing up the above discussion we obtain the following

Theorem 2.

(1) [14] *In order that on $M^{\mathbb{A}}$ there exist an $\mathbb{A}$-affine horizontal distribution it is necessary and sufficient that the Atiyah-Molino class*

$$a(P^V M^{\mathbb{A}}) \in H^{1,1}_{ad\,F}(P^V M^{\mathbb{A}}, \mathfrak{d}_n(\mathbb{A}))$$

vanish.

(2) *If $a(P^V M^{\mathbb{A}}) = 0$, $\mathbb{A}$-affine horizontal distributions are in one-to-one correspondence with elements of $H^{0,1}_{ad\,F}(P^V M^{\mathbb{A}}, \mathfrak{d}_n(\mathbb{A}))$.*

(3) *In order that on $M^{\mathbb{A}}$ there exist an $\mathbb{A}$-affine horizontal distribution it is necessary that for each $Z_0 \in P^V M^{\mathbb{A}}$ there exist a subspace $Q_{Z_0} \subset T_{Z_0} P^V M^{\mathbb{A}}$ which is invariant by α_{P,Z_0}.*

(4) *For existence of an $\mathbb{A}$-affine horizontal distribution it is necessary that there exist a projectable connection in $B^q_{tr} M^{\mathbb{A}}$, that is*

$$a(B^q_{tr} M^{\mathbb{A}}) \in H^{1,1}_{ad\,F}(B^q_{tr} M^{\mathbb{A}}, \mathfrak{g}^q_n)$$

vanish.

Examples. 1. An $\mathbb{A}$-affine connection in the bundle $J^{\mathbb{A}} M$ over a real manifold M is a trivial example of an $\mathbb{A}$-affine horizontal distribution.

2. The quotient group of the additive group $\mathbb{A}^n$ by a discrete subgroup $G \subset \mathbb{A}^n$ generated by a real basis of $\mathbb{A}^n$ is the n-dimensional torus over $\mathbb{A}$ [9]. Since the standard flat connection in $\mathbb{A}^n = J^{\mathbb{A}} \mathbb{R}^n$ is invariant by G, n-dimensional tori admit $\mathbb{A}$-affine horizontal distributions.

Note. As it was shown in [14] in the case of complete manifold $M^{\mathbb{A}}$ an $\mathbb{A}$-affine horizontal distribution is an Ehresmann connection in the sense of R. A. Blumental-J. J. Hebda [15]. Foliations with Ehresmann connections were also studied by N. I. Zhukova in [16].

§4. $\mathbb{A}$-smooth connections on $M^{\mathbb{A}}$

In this section the problem of existence of $\mathbb{A}$-smooth connections in $\mathbb{A}$-smooth principal bundle is considered. Firstly we describe the $\hat{d}$-cohomology [17], which generalizes on the one hand the Dolbeault cohomology of complex manifolds and on the other hand the foliated cohomology developed by I.Vaisman [18] and P.Molino [5].

Let $\Lambda^p M^{\mathbb{A}}$ be the bundle of real exterior p-forms on $M^{\mathbb{A}}$, $\mathbb{A} \otimes \Lambda^p M^{\mathbb{A}}$ the bundle of $\mathbb{A}$-valued p-forms, $\Lambda^p_{\mathbb{A}-lin} M^{\mathbb{A}}$ the subbundle of $\mathbb{A} \otimes \Lambda^p M^{\mathbb{A}}$ consisting of $\mathbb{A}$-linear p-forms, and $\mathbb{A} \otimes \Lambda^1 M^{\mathbb{A}} = \Lambda^1_{\mathbb{A}-lin} M^{\mathbb{A}} \oplus \tilde{\Lambda}^1 M^{\mathbb{A}}$ an arbitrary decomposition into the Whitney sum of $\mathbb{A}$-module bundles (which, for instance, can be constructed with the use of a differentiable nondegenerate $\mathbb{A}$-linear metric on $M^{\mathbb{A}}$). The bundles $\Lambda^{p,q} M^{\mathbb{A}} = \tilde{\Lambda}^p M^{\mathbb{A}} \otimes \Lambda^q_{\mathbb{A}-lin} M^{\mathbb{A}}$ can be treated as subbundles of the bundle $\mathbb{A} \otimes \Lambda^{p+q} M^{\mathbb{A}}$. By $\Omega^{p,q}(U)$, where $U \subset M^{\mathbb{A}}$ is an open subset, we denote the $\mathbb{A}$-module of local sections of $\Lambda^{p,q} M^{\mathbb{A}}$ over U. A form $\theta \in \Omega^{p,q}(U)$ will be called *a (p,q)-form.* The exterior differential $d\theta$ of a (p,q)-form θ lies in $\Omega^{p-1,q+2}(U) \oplus \Omega^{p,q+1}(U) \oplus \Omega^{p+1,q}(U)$, and *the differential* $\hat{d}\theta$ is defined as the component of $d\theta$ belonging to $\Omega^{p+1,q}(U)$. There holds $\hat{d} \circ \hat{d} = 0$, and $H^{p,q} M^{\mathbb{A}}$ denote the homology groups of the complex

$$0 \longrightarrow \Omega^{0,q}(M^{\mathbb{A}}) \xrightarrow{\hat{d}} \Omega^{1,q}(M^{\mathbb{A}}) \xrightarrow{\hat{d}} \Omega^{2,q}(M^{\mathbb{A}}) \xrightarrow{\hat{d}} \dots \,. \tag{10}$$

It follows from definitions that $(0,q)$-form θ is $\mathbb{A}$-smooth if and only if $\hat{d}\theta = 0$.

The Molino's construction [5] of d_F-cohomology of tensorial forms on a foliated principal bundle can be extended to the case of $\mathbb{A}$-smooth principal bundle as follows. Let $P^{\mathbb{A}}(M^{\mathbb{A}}, p, G^{\mathbb{A}})$ be an $\mathbb{A}$-smooth principal bundle over $M^{\mathbb{A}}$ with $\mathbb{A}$-smooth structure Lie group $G^{\mathbb{A}}$, $V^{\mathbb{A}}$ an $\mathbb{A}$-module isomorphic to $\mathbb{A}^k$ for some k, $GL_{\mathbb{A}}(V^{\mathbb{A}})$ the group of $\mathbb{A}$-linear transformations of $V^{\mathbb{A}}$, and $\rho : G^{\mathbb{A}} \to GL_{\mathbb{A}}(V^{\mathbb{A}})$ an $\mathbb{A}$-smooth representation of $G^{\mathbb{A}}$ in $V^{\mathbb{A}}$. An r-form θ with values in $V^{\mathbb{A}}$ is called *tensorial* [19] if $R_g^*\theta = \rho(g^{-1})\theta$ for $g \in G^{\mathbb{A}}$, and $\theta(v_1, \dots, v_r) = 0$ if at least one of the vectors $v_1, \dots, v_r$ is vertical. A tensorial $(p+q)$-form θ with values in $V^{\mathbb{A}}$ is said to be *a (p,q)-form* if for every local section $s : U \subset M^{\mathbb{A}} \to P^{\mathbb{A}}$ $s^*\theta$ is a (p,q)-form on $M^{\mathbb{A}}$. Let $\{s_\alpha : U_\alpha \subset M^{\mathbb{A}} \to P^{\mathbb{A}}\}_{\alpha \in A}$ be a family of $\mathbb{A}$-smooth sections where $\{U_\alpha\}_{\alpha \in A}$ is a covering of $M^{\mathbb{A}}$. We have $s_\beta = R_{g_{\beta\alpha}} \circ s_\alpha$ where $g_{\beta\alpha} : U_\alpha \cap U_\beta \to G^{\mathbb{A}}$ are $\mathbb{A}$-smooth functions. Denote $\theta_\alpha = s_\alpha^*\theta$. Then we have

$$\theta_\beta = \rho(g_{\beta\alpha}^{-1})\theta_\alpha. \tag{11}$$

Tensorial (p,q)-forms with values in $V^{\mathbb{A}}$ on $P^{\mathbb{A}}$ are in one-to-one correspondence with families $\{\theta_\alpha\}_{\alpha \in A}$ of local $V^{\mathbb{A}}$-valued forms on $M^{\mathbb{A}}$ satisfying (11). Since $\rho(g_{\beta\alpha}^{-1})$ is $\mathbb{A}$-smooth, from (11) it follows that $\hat{d}\theta_\beta = \rho(g_{\beta\alpha}^{-1})\hat{d}\theta_\alpha$. Hence the family $\{\hat{d}\theta_\alpha\}_{\alpha \in A}$ determines a tensorial $(p+1,q)$-form, which will be denoted by $\hat{d}\theta$. Thus we obtain the operator

$$\hat{d} : \Omega_\rho^{p,q}(P^{\mathbb{A}}, V^{\mathbb{A}}) \to \Omega_\rho^{p+1,q}(P^{\mathbb{A}}, V^{\mathbb{A}})$$

where $\Omega_\rho^{p,q}(P^{\mathbb{A}}, V^{\mathbb{A}})$ is the $\mathbb{A}$-module of tensorial (p,q)-forms with values in $V^{\mathbb{A}}$. The corresponding cohomology groups will be denoted by $H_\rho^{p,q}(P^{\mathbb{A}}, V^{\mathbb{A}})$.

Let now Γ be a connection in $P^{\mathbb{A}}$ and

$$\omega : TP^{\mathbb{A}} \to \mathfrak{g}^{\mathbb{A}} \tag{12}$$

the connection form of Γ. If the mapping $\omega_Z : T_Z P^{\mathbb{A}} \to \mathfrak{g}^{\mathbb{A}}$ is $\mathbb{A}$-linear for each $Z \in P^{\mathbb{A}}$, then Γ is called an *adapted* connection, and if the mapping (12) is $\mathbb{A}$-smooth, Γ is called an $\mathbb{A}$-*smooth* connection. Repeating word for word the discussion from the section 2.6 of [5] we obtain that the $(1,1)$-component $\Omega_{1,1}$ of the curvature form of an adapted connection is $\hat{d}$-closed, the class $a(P^{\mathbb{A}}) = [\Omega_{1,1}] \in H_{ad}^{1,1}(P^{\mathbb{A}}, \mathfrak{g}^{\mathbb{A}})$, where ad is the adjoint representation of $G^{\mathbb{A}}$ in $\mathfrak{g}^{\mathbb{A}}$, does not depend on a choice of adapted connection, and there holds

Theorem 3. *$a(P^{\mathbb{A}}) = 0$ if and only if there exists an $\mathbb{A}$-smooth connection in $P^{\mathbb{A}}$.*

The class $a(P^{\mathbb{A}})$ is called *the Atiyah class of $P^{\mathbb{A}}$*.

Our next aim is to establish relations between the $\hat{d}$-cohomology and the d_F-cohomology with respect to the canonical $\mathring{\mathbb{A}}$-foliation. Let $TM^{\mathbb{A}} = T^v M^{\mathbb{A}} \oplus T^h M^{\mathbb{A}}$ be the decomposition of the tangent bundle of $M^{\mathbb{A}}$ into the Whitney sum of the submodule $T^v M^{\mathbb{A}}$ formed by tangent vectors to leaves of the canonical $\mathring{\mathbb{A}}$-foliation and some complementary subbundle $T^h M^{\mathbb{A}}$. This decomposition induces the decomposition of the cotangent bundle $\Lambda^1 M^{\mathbb{A}} = \Lambda_v^1 M^{\mathbb{A}} \oplus \Lambda_h^1 M^{\mathbb{A}}$. If $\{e_a\} = \{e_0, e_1, \dots, e_m\}$ is a basis in $\mathbb{A}$ such that $e_0 = 1, e_{\hat{a}} \in \mathring{\mathbb{A}}$ where $\hat{a} = 1, \dots, m$, and $X^i = x^{ia}e_{ia} = x^i e_0 + x^{i\hat{a}}e_{\hat{a}}$ are the expansions of local $\mathbb{A}$-coordinates on $M^{\mathbb{A}}$ in terms of the basis $\{e_a\}$, then $T^v M^{\mathbb{A}}$ is spanned by $\partial/\partial x^{i\hat{a}}$, $T^h M^{\mathbb{A}}$ by $\partial/\partial x^i - \mu_i^{j\hat{a}}\partial/\partial x^{j\hat{a}}$, $\Lambda_v^1 M^{\mathbb{A}}$ by $\eta^{i\hat{a}} = dx^{i\hat{a}} + \mu_j^{i\hat{a}}dx^j$, and $\Lambda_h^1 M^{\mathbb{A}}$ by dx^i. Let, in addition, the subbundle $\tilde{\Lambda}^1 M^{\mathbb{A}}$ is spanned by the forms $\theta^{i\hat{a}} = dx^{i\hat{a}} + \lambda_j^{i\hat{a}}dX^j$. We define the following two mappings $\beta_1 : \Lambda^1_{\mathbb{A}-lin}M^{\mathbb{A}} \ni \omega \mapsto \bar{\omega} \in \Lambda_h^1 M^{\mathbb{A}}$ and $\beta_2 : \tilde{\Lambda}^1 M^{\mathbb{A}} \ni \tilde{\omega} \mapsto \hat{\omega} \in \Lambda_v^1 M^{\mathbb{A}}$ as follows: $\bar{\omega} = \pi_0^q \circ \omega$ and $\hat{\omega} = \pi_0^q \circ \tilde{\omega} \circ pr_v$ where $pr_v : TM^{\mathbb{A}} \to T^v M^{\mathbb{A}}$ is the projection. In local coordinates there hold $f_i dX^i \mapsto f_i^0 dx^{i0}$ and $f_{i\hat{a}}\theta^{i\hat{a}} \mapsto f_{i\hat{a}}^0 \eta^{i\hat{a}}$ where $f_i = f_i^a e_a$ and $f_{i\hat{a}} = f_{i\hat{a}}^b e_b$ are the expansions in terms of the basis $\{e_a\}$. The mappings β_1 and β_2 induce the mappings $\beta : \Lambda^{p,q}M^{\mathbb{A}} \to \Lambda_v^p M^{\mathbb{A}} \otimes \Lambda_h^q M^{\mathbb{A}} = \Lambda_{fol}^{p,q}M^{\mathbb{A}}$ and $\beta : \Omega^{p,q}M^{\mathbb{A}} \to \Omega_{fol}^{p,q}M^{\mathbb{A}}$ where $\Omega_{fol}^{p,q}M^{\mathbb{A}}$ is the vector space of smooth sections of the bundle $\Lambda_{fol}^{p,q}M^{\mathbb{A}}$. It can be easily verified that $d_F \circ \beta = \beta \circ \hat{d}$, and so β is a morphism of the complex (10) into the foliated complex

$$0 \longrightarrow \Omega_{fol}^{0,q}(M^{\mathbb{A}}) \xrightarrow{d_F} \Omega_{fol}^{1,q}(M^{\mathbb{A}}) \xrightarrow{d_F} \Omega_{fol}^{2,q}(M^{\mathbb{A}}) \xrightarrow{d_F} \dots .$$

Then we obtain the corresponding morphism of the cohomology groups

$$\beta^* : H^{p,q}M^{\mathbb{A}} \to H_{fol}^{p,q}M^{\mathbb{A}}.$$

This morphism β can be extended to the case of tensorial forms on principal bundles in the following situation. Suppose the leaf $\overset{\circ}{G}{}_e^{\mathbb{A}}$ through the identity element e of the structure group $G^{\mathbb{A}}$ of an $\mathbb{A}$-smooth principal bundle $P^{\mathbb{A}}(M^{\mathbb{A}}, p, G^{\mathbb{A}})$ is a closed subgroup in $G^{\mathbb{A}}$, and let $\pi : G^{\mathbb{A}} \to G = G^{\mathbb{A}}/\overset{\circ}{G}{}_e^{\mathbb{A}}$ be the canonical epimorphism. Then we can factorize $P^{\mathbb{A}}$ by the action of $\overset{\circ}{G}{}_e^{\mathbb{A}}$ and obtain a quotient foliated principal bundle $P_{tr}(M^{\mathbb{A}}, p_{tr}, G)$. Let $\pi_P : P^{\mathbb{A}} \to P$ denotes the canonical projection. An $\mathbb{A}$-module $V^{\mathbb{A}}$, isomorphic to $\mathbb{A}^k$, can be factorized by the submodule corresponding to $\overset{\circ}{\mathbb{A}}{}^k \subset \mathbb{A}^k$, let V be the corresponding real vector space, and $\pi_V : V^{\mathbb{A}} \to V$ the canonical ephimorphism (of vector spaces). In this case a representation $\rho : G^{\mathbb{A}} \to GL_{\mathbb{A}}(V^{\mathbb{A}})$ induces a representation $\rho_{tr} : G \to GL_{\mathbb{R}}(V)$, and to each tensorial (p,q)-form θ determined by a family of local $V^{\mathbb{A}}$-valued (p,q)-forms $\{\theta_\alpha\}$ we assign the form θ_{tr} determined by the family of foliated sections $\{\pi_P \circ s_\alpha\}_{\alpha \in A}$ and the family of local V-valued (p,q)-forms $\{\theta_{tr,\alpha} = \pi_V \circ \theta_\alpha\}$. Thus we obtain the morphism of complexes $\beta : \Omega_\rho^{p,q}(P^{\mathbb{A}}, V^{\mathbb{A}}) \to \Omega^{p,q}_{\rho_{tr},F}(P_{tr}, V)$ and the corresponding morphism of cohomology groups $\beta^* : H_\rho^{p,q}(P^{\mathbb{A}}, V^{\mathbb{A}}) \to H^{p,q}_{\rho_{tr},F}(P_{tr}, V)$.

Since, by definition, the connection form (12) of an adapted connection Γ in $P^{\mathbb{A}}$ is $\mathbb{A}$-linear on $T_Z P^{\mathbb{A}}$, there is a unique connection Γ_{tr} in P_{tr} with the connection form $\omega_{tr} : TP_{tr} \to \mathfrak{g}$ determined by the relation $\omega_{tr} \circ T\pi_P = \pi_{\mathfrak{g}} \circ \omega$. Analyzing the structure equations of the connections ω and ω_{tr} (see [5], §2.6) and taking into account the structure of $\mathbb{A}$-smooth mappings (see (2)) we obtain

Theorem 4.

(1) *If a connection* Γ *in* $P^{\mathbb{A}}$ *is* $\mathbb{A}$*-smooth, then the connection* Γ_{tr} *in* P_{tr} *is projectable.*
(2) $\beta^*(\alpha(P^{\mathbb{A}})) = \alpha(P_{tr})$.

Corollary.

(1) *In order that there exists an* $\mathbb{A}$*-smooth connection in* $P^{\mathbb{A}}$ *it is necessary that* $\alpha(P_{tr}) = 0$.
(2) *An* $\mathbb{A}$*-smooth* $\mathbb{A}$*-linear connection* $\nabla^{\mathbb{A}}$ *on an* $\mathbb{A}$*-smooth manifold* $M^{\mathbb{A}}$ *induces a transversal* [5] *connection* ∇_{tr} *on* $M^{\mathbb{A}}$ *with respect to the canonical* $\mathring{\mathbb{A}}$*-foliation.*

Notes. In the same way as in Section 2 taking compositions of transition functions of an $\mathbb{A}$-smooth bundle $P^{\mathbb{A}}$ along loops lying in a leaf L of the canonical $\mathring{\mathbb{A}}$-foliation on $M^{\mathbb{A}}$ through $X_0 \in L$ we obtain a homomorphism α of $\pi_1(L, X_0)$ into the group of germs of automorphisms of $P^{\mathbb{A}}$ over a neighborhood of X_0, and so in order that there exist an $\mathbb{A}$-smooth connection in $P^{\mathbb{A}}$ it is necessary that there exist a right-invariant horizontal distribution along L which is invariant under this representation. In the case when $P^{\mathbb{A}}$ is a natural bundle [4] the homomorphism α is a holonomy representation, therefore the holonomy of $M^{\mathbb{A}}$ may be an obstruction for existence of $\mathbb{A}$-smooth connections in $\mathbb{A}$-smooth natural bundles over $M^{\mathbb{A}}$.

Applying the Weil functor [4] $J^{\mathbb{A}}$ to a real principal bundle $P(M, p, G)$ we obtain, evidently, an $\mathbb{A}$-smooth principal bundle, which admits $\mathbb{A}$-smooth connections.

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