DIFFERENTIAL GEOMETRY AND APPLICATIONS
Proc. Conf., Aug. 28 – Sept. 1, 1995, Brno, Czech Republic
Masaryk University, Brno 1996, 321–328

SYMMETRIES OF LAGRANGIAN FIELDS ARE HOMOTHETIES IN THE HOMOGENEOUS CASE

János Szenthe

DEFINITIONS. Let M be a smooth manifold, $\mathcal{O}(M)$ be the zero section of the tangent bundle TM and $\check{TM} = TM - \mathcal{O}(M)$. Let $X : \check{TM} \to TTM$ be a 2nd-order differential equation over M, a diffeomorphism $\varphi : M \to M$ is said to be a *symmetry of the 2nd-order differential equation* X if

$$(T\varphi)_* X = TT\varphi \circ X \circ T\varphi^{-1} = X$$

holds. Consider the canonical projection $\pi_M : TM \to M$ and an integral curve $\xi : I \to TM$ of X, then $\pi_M \circ \xi : I \to M$ is called a *base integral curve* of X. A diffeomorphism $\varphi : M \to M$ is a symmetry of X if and only if it maps the set of base integral curves of X onto itself. In the particular case when X is a *spray*, in other words, when $\mathcal{L}_A X = X$ holds, where $A : TM \to TTM$ is the Liouville field on TM, then φ is called a *symmetry of the spray*. Subsequently, the case is considered when X is obtained as a *Lagrangian field* of a Lagrangian function $L : TM \to \mathbf{R}$, which means that X satisfies the Euler-Lagrange Equation

$$\iota_X dd_v = d(L - AL)$$

where d_v is the vertical differential (see e.g. [Be] pp 14-28 and [Go] pp 169-177). In such a case (M, L, X) is called a Lagrangian system which is said to be *regular* if the 2-form $dd_v L$ is regular. If L is homogeneous of degree k, in other words, if $AL = kL$ holds then (M, L, X) is said to be *homogeneous of degree* k. If (M, L, X) is a regular homogeneous Lagrangian system then X is a spray (see e.g. [Sz1]).

In particular, let (M, g) be a semi-Riemannian manifold and $X : TM \to TTM$ the spray associated with its Levi-Cività covariant derivation ∇. Then X is also a Lagrangian field associated with the Lagrangian function $L : v \mapsto g(v, v),\ v \in TM$ (see e.g. [Be] pp 28-32). Moreover, the symmetries of X are affine transformations

1991 *Mathematics Subject Classification*. Primary 58F05; secondary 70H35, 53C60.
Key words and phrases. Sprays, Lagrangian systems, symmetries of Lagrangian systems.
Supported by Hungarian Nat. Found. for Sci. Research Grant No. T17314(1995).
This paper is in final form and no version of it will be submitted for publication elsewhere.

of (M, g). In fact, if $\psi : M \to M$ is an affine transformation of (M, g) then ψ preserves the set of geodesics of ∇ which are the base integral curves of X and consequently ψ is a symmetry of X. Conversely, if a diffeomorphism $\varphi : M \to M$ is a symmetry of X, then φ is an affine transformation of (M, g); in fact, a covariant derivation ∇' is defined over M by

$$\nabla'_U V = T\varphi^{-1} \nabla_{T\varphi U} T\varphi V, \; U, V \in \mathcal{T}(TM),$$

which has the same geodesics as ∇ and has also zero torsion; but then $\nabla = \nabla'$ holds; which means that ψ is an affine transformation of (M, g). It is a basic result in Riemannian geometry, that any affine transformation of an irreducible Riemannian manifold is a homothety. An extension of the above result to homogeneous Lagrangian systems is presented below.

DEFINITION. Let (M, L, X) be a Lagrangian system over a smooth manifold M. A diffeomorphism $\varphi : M \to M$ is said to be a *homothety* of the Lagranian system if

$$(T\varphi)^* L = L \circ T\varphi = \lambda \cdot L$$

holds with some $\lambda \in \mathbf{R}$. In the particular case when L is the energy function of a semi-Riemannian manifold (M, g), in other words, when $L(v) = g(v, v), \; v \in TM$ holds, and a diffeomorphism $\psi : M \to M$ is a usual homothety of the semi-Riemannian manifold, in other words, if

$$g(T\psi U, T\psi V) = \mu g(U, V)$$

holds with some $\mu \in \mathbf{R}$ for $U, V \in \mathcal{T}(M)$, then also $((T\psi)^*)L(W) = L(T\psi(W)) = g(T\psi(W), T\psi(W)) = \mu g(W, W) = \mu L(W)$ holds for $W \in \mathcal{T}(M)$, which shows that ψ is a homothety of the corresponding Lagrangian system (M, L, X) as well. Assume conversely that $\varphi : M \to M$ is a homothety of the Lagrangian system (M, L, X) then $g(T\varphi(W), T\varphi(W)) = ((T\phi)^* L)(W) = \lambda \cdot L(W) = \lambda \cdot g(W, W)$ holds for any $W \in \mathcal{T}(M)$; but then $g(T\varphi U, T\varphi V) = \lambda \cdot g(U, V)$ is obviously valid for any $U, V \in \mathcal{T}(M)$ which implies that φ is a usual homothety of the semi-Riemannian manifold (M, g) too. A homothety of a regular Lagrangian system obviously yields a symmetry of its Lagrangian field as well.

It is a fundamental fact that a 2nd-order differential equation $X : \breve{TM} \to TTM$ defines a connection by the definition $\gamma = -\mathcal{L}_X v$ where v is the vertical endomorphism (see e.g. [K]). The connection form γ defines a derivation ι_γ of degree 0 of the algebra $\wedge(\breve{TM})$ by the definition:

$$\iota_\gamma \xi : (U_1, ..., U_k) \mapsto \sum_i \xi(U_1, ..., U_{i-1}, \gamma U_i, U_{i+1}, ..., U_k)$$

where $\xi \in \wedge^k(\breve{TM})$. But then ι_γ yields a derivation d_γ of degree 1 by the definition $d_\gamma = [\iota_\gamma, d]$ as obvious calculations show.

LEMMA 1. Let $X : T\breve{M} \to TTM$ be a 2nd-order differential equation over a smooth manifold M and $\gamma = -\mathcal{L}_X \upsilon$ the connection form defined by X. Then

$$d_\gamma = [d_\upsilon, \mathcal{L}_X]$$

holds where d_υ is the vertical differential over the smooth manifold M.

Proof. It is sufficient to show that the differential operators d_γ and $[d_\upsilon, \mathcal{L}_X]$ have the same effect on any 0-form $\xi = F$ and 1-form $\xi = dF$ where $F \in \mathcal{F}(M)$ (see e.g. [Go] pp 87-89).

1st case: $\xi = F$. Consider a smooth vector field $V \in \mathcal{T}(T\breve{M})$, then the following hold

$$(d_\gamma)(v) = dF(\gamma V) = dF(-\mathcal{L}_X \upsilon V) =$$
$$= -dF([X, \upsilon V] - \upsilon[X, V]) = -X((\upsilon V)F) + (\upsilon V)(XF) + \upsilon[X, V]F.$$

On the other hand

$$([d_\upsilon, \mathcal{L}_X]F)V = (d_\upsilon(XF) - \mathcal{L}_X \iota_\upsilon dF)V =$$
$$= (\upsilon V)(XF) - \iota_V \mathcal{L}_X \iota_\upsilon dF =$$
$$= (\upsilon V)(XF) - X((\upsilon V)F) + (\iota_\upsilon dF)([X, V]).$$

Thus the identity is valid in the case considered.

2nd case: $\xi = dF$. The following are valid now:

$$d_\gamma dF = (\iota_\gamma d - d\iota_\gamma)dF = -d\iota_\gamma dF =$$
$$= -d(\iota_\gamma d - d\iota_\gamma)F = -dd_\gamma F,$$
$$[d_\upsilon, \mathcal{L}_X]dF = (d_\upsilon \mathcal{L}_X - \mathcal{L}_X d_\upsilon)dF =$$
$$= d_\upsilon d\mathcal{L}_X + \mathcal{L}_X dd_\upsilon F == -d(d_\upsilon \mathcal{L}_X - \mathcal{L}_X d_\upsilon)F =$$
$$= -d[d_\upsilon, \mathcal{L}_X]F = [d_\upsilon, \mathcal{L}_X]dF.$$

Thus the validity of the identity in the second case follows now by that of the first one. ♠

The connection γ yields the horizontal and vertical projection fields $\chi, \varpi : TT\breve{M} \to TT\breve{M}$ such that $\gamma = \chi - \varpi$ holds (see e.g. [K]).

LEMMA 2. Let (M, L, X) be a regular Lagrangian system over a smooth manifold M such that L is a first integral of X. If $\chi : TTM \to TTM$ is the horizontal projection field associated with the connection form $\gamma = -\mathcal{L}_X \upsilon$, then $d_\chi L = 0$ holds for the antiderivation $d_\chi = [\iota_\chi, d] = \iota_\chi d - d\iota_\chi$ associated with χ.

Proof. Let $F : T\breve{M} \to \mathbf{R}$ be an arbitrary smooth function. Then considering the above expression of the horizontal projection field, the preceding Lemma and the Cartan Identity, the following is obtained

$$2d_\chi F = (d + d_\gamma)F = dF + [d_\upsilon, \mathcal{L}_X]F =$$

$$= dF + d_\upsilon(d\iota_X + \iota_X d)F - (d\iota_X + \iota_X d)d_\upsilon F =$$

$$= dF + d_\upsilon \iota_X dF - d(AF) - \iota_X dd_\upsilon F =$$

$$= d(F - AF) - \iota_X dd_\upsilon F + d_\upsilon(XF),$$

where A is the Liouville field and the equality $\upsilon X = A$ was also applied. Therefore, in the particular case when $F = L$ the above identity and the Euler-Lagrange Equation yield that

$$2d_\chi L = d_\upsilon(XL)$$

holds. Since L is a first integral of X, the assertion of the Lemma directly follows. ♠

COROLLARY. (J. Klein) *Let (M, L, X) be a regular Lagrangian system which is homogeneous of degree k where $k \neq 1$ then $d_\chi L = 0$ holds where χ is the horizontal projection field associated with the connection $\gamma = -\mathcal{L}_X \upsilon$.*

Proof. In fact, the Euler Lagrange Equation has now the following form

$$\iota_X dd_\upsilon L = (1 - k)dL$$

and therefore $(1 - k)\iota_X dL = (1 - k)XL = 0$ holds and thus $XL = 0$ is obtained. But then the preceding Lemma applies. ♠

LEMMA 3. Let (M, L, X) be a regular Lagrangian system such that L is homogeneous of degree k where $k \neq 1$. Then (M, L^p, X) is a regular Lagrangian system as well for any $p > 0$.

Proof. In fact, the following equalities are obviously valid:

$$dd_\upsilon(L^p) = p((p-1)L^{p-2}dL \wedge d_\upsilon L + L^{p-1}dd_\upsilon L),$$

$$\iota_X dd_\upsilon(L^p) = p(p-1)L^{p-2}(\iota_X dL \cdot d_\upsilon L - \iota_X d_\upsilon L \cdot dL) + pL^{p-1}\iota_X dd_\upsilon L =$$

$$= -p(p-1)kL^{p-1}dL + pL^{p-1}\iota_X dd_\upsilon L,$$

$$d(L^p - A(L^p)) = pL^{p-1}dL - kp^2L^{p-1}dL =$$

$$= -p(p-1)kL^{p-1}dL + p(1-k)L^{p-1}dL.$$

But then the assertion of the Lemma directly follows. ♠

The assertion of the preceding Lemma also follows from Proposition 1 of [Sz1] where the assumption that L, L' are first integrals of X was omitted.

DEFINITIONS. Let $X : TM \to TTM$ be a spray over a smooth manifold M then the corresponding connection form $\gamma = -\mathcal{L}_X \upsilon$ is homogeneous (see e.g. [K]). Then the *curvature field* R of the connection γ is defined by

$$R(U, V) = -\varpi[\chi U, \chi V], \; U, V \in \mathcal{T}(T\check{M}).$$

If $v \in T_z M$ is a fixed tangent vector then the subspace $R_v \subset \mathcal{V}(TM)$ spanned by the vectors $R(U,V)_v$, $U,V \in \mathcal{T}(T\breve{M})$ is called the *image* of the curvature tensor at v. If the dimension of the subspaces R_v, $v \in T_z\breve{M}$ is a constant function on $T_z\breve{M}$ then its value r_z is called the *rank* of X at $z \in M$ (see [K]).

PROPOSITION 4. (J. Klein) *If the spray X is Lagrangian field of a Lagrangian function $L : TM \to \mathbf{R}$ which is homogeneous of degree $k \neq 1$ and such that its rank r_z exists for $z \in M$, then $r_z \leq m-1$ holds where $m = dim M$.*

Proof. In fact, it is sufficient to show that $dL(R(U,V)) = 0$ holds for any $U,V \in \mathcal{T}(T\breve{M})$. Actually, since $d_\chi L = 0$ holds by the Corollary to Lemma 2, the following are valid too

$$dL(R(U,V))) = dL(\varpi[\chi U, \chi V]) =$$

$$= dL([\chi U, \chi V] - \chi[\chi U, \chi V]) = dL([\chi U, \chi V]) =$$

$$= (\chi U)((\chi V)L) - (\chi V)((\chi U)L) = 0.$$

But then the assertion follows by the observation above. ♠

DEFINITIONS. Let $X : TM \to TTM$ be a spray over a smooth manifold M and $\gamma = -\mathcal{L}_X v$ the corresponding connection form which is homogeneous. The connection form γ defines the *horizontal distribution*

$$\mathcal{H} = \{\mathcal{H}_v | v \in TM\}$$

where $\mathcal{H}_v = \{w \in T_v TM | \varpi(w) = 0\}$ and ϖ is the vertical projection corresponding to the connection form γ (see e. g. [K]). The horizontal distribution $\mathcal{H}$ in turn defines for each piecewise smooth loop λ based at $z \in M$ the *parallel translation* $\pi_\lambda : T_z M \to T_z M$ along λ which proves to be a homogeneous diffeomorhism of $T_z\breve{M}$ (see [Sz2]). The above parallel translations form the *holonomy group* Φ_z defined by the distribution at z.

PROPOSITION 5. Let (M,L,X) be a homogeneous and regular Lagrangian system such that the rank of X exits and has maximal value everywhere on M. Then the orbits of the holonomy group Φ_z of X are codimension 1 submanifolds of $T_z\breve{M}$ for any $z \in M$.

The proof of the above Proposition is given in [Sz2].

THEOREM 6. Let M be smooth m-dimensional manifold and (M,L,X) a regular Lagrangian system where L is homogeneous of degree k with $k \neq 0,1$ and the rank of X exits and has maximal value everywhere on M. Then any diffeomorphism $\varphi : M \to M$ which yields a symmetry of X is a homothety of L as well.

Proof. Observe first that there is no loss of generality by assuming that L is everywhere non-negative on TM. In fact, assume that the Theorem is true for sytems with everywhere non-negative Lagrangians and consider an arbitrary

Lagrangian system (M, L, X) which satisfies the conditions of the Theorem. Then by Lemma 3 a Lagrangian system (M, L^2, X) is obtained which also satisfies the conditions of the Theorem and which is homogeneous of degree $2k$. If $\varphi : M \to M$ yields a symmetry of X, then φ is a homothety of L^2 by the assumption. In other words

$$((T\varphi)^* L(v))^2 = ((T\varphi)^* L^2)(v) = \lambda \cdot L^2(v) = (\sqrt{\lambda} L(v))^2$$

holds for $v \in TM$. But then φ is obviously a homothety of L as well.

Observe secondly that all the values of L on $\check{T}M$ are regular. In fact the Euler-Lagrange Equation

$$\iota_X dd_v L = (1-k) dL$$

and the regularity of L yield that X has to vanish at a sigular point of L. But X being a 2nd-order differential equation cannot vanish on $\check{T}M$.

Consider now a Lagrangian system which satisfies the conditions of the Theorem and where the Lagrangian L is everywhere non-negative. Then take first a smooth loop $\xi : [0,1] \to M$ with $\xi(0) = \xi(1) = z \in M$, a tangent vector $v \in T_z\check{M}$ and the vector $v' = \pi_\xi v \in T_z M$ obtained by the parallel translation π_ξ of v along ξ. It will be shown first that

$$L(v) = L(v')$$

is valid. In fact, consider the horizontal projection field χ associated with the connection γ and let $\hat{\xi} : [0,1] \to TM$ be the corresponding horizontal lift of the loop ξ with $\hat{\xi}(0) = v$. Then $\hat{\xi}(1) = v'$ holds; moreover,

$$d(L \circ \hat{\xi}) = dL(\dot{\hat{\xi}}) = dL(\chi\dot{\hat{\xi}}) = d_\chi L(\dot{\hat{\xi}}) = 0$$

holds, since $d_\chi L = 0$ is valid according to the Corollary to Lemma 2. Consequently, the following holds as well:

$$L(v') = L \circ \hat{\xi}(1) = L \circ \hat{\xi}(0) = L(v).$$

Assume now that the smooth loop ξ is embedded into a smooth family $\xi_\theta,\ \theta \in [0,1]$ of smooth loops; more precisely, consider a smooth map

$$\Xi : [0,1] \times [0,1] \to M$$

such that for each fixed $\theta \in [0,1]$ a smooth loop $\xi_\theta(\tau) = \Xi(\tau, \theta),\ \tau \in [0,1]$ with $\xi_\theta(0) = \xi_\theta(1) = z$ is obtained, moreover $\xi_0(\tau) = z$ for $\tau \in [0,1]$ and $\xi_1 = \xi$. Let now $v'_\theta = \pi_{\xi_\theta} v$ be the vector obtained by parallel translation of v along ξ_θ where $\theta \in [0,1]$. Then a smooth curve

$$\theta \mapsto v'_\theta \in T_z M,\ \theta \in [0,1]$$

is obtained, where $v'_0 = v$ and $v'_1 = v'$. Moreover, the above curve lies in that level set L_v of the function L which contains v by the preceding observation. Consequently, the tangent vector

$$\frac{d}{d\theta} v'_\theta|_{\theta=0} \in T_v T_z M$$

of the above curve is in the tangent space $T_v L_v$ of the level set L_v which is a submanifold. The preceding observations obviously extend to the case of piecewise smooth loops. On the other hand, the orbit $\Phi_z(v)$ of the holonomy group is a codimension 1 submanifold of $T_z\breve{M}$ by Proposition 5, and therefore any tangent vector of the orbit $\Phi_z(v)$ at v is obtainable as a linear combination of tangent vectors of curves of the above kind. Therefore, the inclusion

$$T_v \Phi_z(v) \subset T_v L_v$$

holds. Furthermore, for any horizontal smooth curve $\eta : I \to TM$ the equality $\dot{\eta} L = d_\chi L(\dot{\eta}) = 0$ is valid by Lemma and consequently even

$$\mathcal{H}_v \subset T_v L_v$$

holds since the level set L_v is a smooth codimension 1 submanifold for $v \in T\breve{M}$. But then

$$2m - 1 = dim T_v L_v \geq dim(\mathcal{H}_v \oplus T_v \Phi_z(v)) \geq 2m - 1$$

holds since $T_v \Phi_z(v)$ is included in the vertical subspace $\mathcal{V}_v$ and since $dim T_v \Phi_z(v) = m - 1$ by assumptions of the theorem. But then

$$T_v L_v = \mathcal{H}_v \oplus T_v \Phi_z(v)$$

is valid in consequence of the preceding inequalities.

By basic observations $(M, (T\varphi)^* L, (T\varphi)_* X)$ is a regular Lagrangian system which is homogeneous of degree k as well. Moreover, $(T\varphi)_* X = X$ holds by assumption. Consequently, the preceding observations applied to the Lagrangian system $(M, (T\varphi)^* L, X)$ yield that

$$T_v (T\varphi^* L)_v = \mathcal{H}_v \oplus T_v \Phi_z(v)$$

holds, provided that $v \subset T\breve{M}$ is valid. Therefore, the equality

$$T_v L_v = T_v((T\varphi)^* L)_v$$

holds. But then the regular level sets of L and $(T\varphi)^* L$ coincide.

Consider now the restricted functions $L\lceil T_z M$ and $(T\varphi)^* L\lceil T_z M$ in case of a $z \in M$. The above functions cannot have the value 0 on $\breve{T}M$; in fact, since the orbits of the holonomy group are included in level sets of the above functions and these orbits intersect each 1-dimensional subspace of $T_z M$, the homogeneity of the above functions implies that the value 0 can be only singular; but the above functions cannot have singular value on $T\breve{M}$ by a former observation. Since both the functions $L\lceil T_z M$ and $(T\varphi)^* L\lceil T_z M$ are homogeneous of degree $k \neq 0$ and everywhere non-negative, a ray $\mathbf{R}^+ v,\ v \in T_z\breve{M}$ intersects each level set of L and of $(T\varphi)^*L$ in a strictly monotone way. But then there is a function $\lambda_z : \mathbf{R}^+ \to \mathbf{R}^+$ such that

$$((T\varphi)^*L)\lceil T_zM = \lambda_z \circ (L\lceil T_zM)$$

holds and the function λ_z is smooth. But then by homogeneity of the restricted functions

$$\xi^k\lambda_z(L(v)) = \xi^k((T\varphi)^*L(v)) = ((T\varphi)^*(\xi^kL(v)) = (T\varphi)^*L)(\xi v) =$$

$$= \lambda_z(L(\xi v)) = \lambda_z(\xi^kL(v))$$

holds for any $\xi \in \mathbf{R}$ which implies that the function λ_z is homogeneous of degree 1. In other words,

$$\lambda_z(\tau) = \sigma(z)\cdot\tau,\;\; \tau \in \mathbf{R}$$

holds with some $\sigma(z) \in \mathbf{R}$ and consequently, $((T\varphi)^*L)(v) = L(\sigma(z)v),\;\; v \in T_zM$ is valid too.

Considering the definition of a homothety of L, it is sufficient now to show that the function $z \mapsto \sigma(z),\;\; z \in M$ is constant. In fact, consider the pull back $\tilde{\sigma} : TM \to \mathbf{R}$ of the above function by π_M. Since L and $(T\varphi)^*L$ have the same level sets the function $\tilde{\sigma}$ is constant on each of these common level sets. But then $\tilde{\sigma}$ itself must be constant. Consequently, σ is constant as well. ♠

References

[Be] Besse, A., *Manifolds all of whose Geodesics are Closed.* Berlin-Heidelberg-New York, 1978

[Go] Godbillon, C., *Géométrie Différentielle et Mécanique Analytique.* Paris, 1969

[Kl] Klein, J., Geometry of sprays, *Proc. of the IUTAM-ISIMM Symposium of Torino*, 177-193

[Sz1] Szenthe, J., Lagrangians and sprays, *Ann. Univ. Sci. Budapest* **35**(1992), 103-107

[Sz2] Szenthe, J., On the holonomy groups of homogeneous connections. *(To appear)*

Department of Geometry
Eötvös University, Budapest
Rákóczi út 5, H-1088, HUNGARY