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# PROJECTORS OF A 3–WEB

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Abstract. Our aim is to investigate in details projectors of a 3-web, to find a correspondence between Nijenhuis brackets (denoted by [ , ]) of the projector fields on a 3-web manifold and properties of the Chern connections corresponding to the web.

**1. Three-webs.** A 3-web on an even-dimensional manifold $M_{2n}$ can be regarded as a triple $(D_1, D_2, D_3)$ of smooth $n$-dimensional integrable distributions which are pairwise complementary ($TM = D_\alpha \oplus D_\beta$ for $\alpha \neq \beta$). Web equivalences are local diffeomorphisms of the underlying manifolds $f : M \to M'$ which respect web distributions, $Tf(D_\alpha) = D'_\alpha$ (another speaking, the tangent map $Tf$ is interchangeable with projector tensor fields).

The web distributions determine three couples of complementary projectors $P^\beta_\alpha$, $\ker P^\alpha_\beta = \operatorname{im} P^\beta_\alpha = D_\alpha$, $P^\beta_\alpha + P^\alpha_\beta = I$, $(P^\beta_\alpha)^2 = P^\beta_\alpha$, and $(1,1)$-tensor fields $B_\gamma = P^\alpha_\gamma - P^\gamma_\beta$ which are involutory, $(B_\gamma - I)(B_\gamma + I) = 0$. Since all web distributions are integrable they form integrable almost product structures $[D_\alpha, D_\beta]$ for which $[P^\beta_\alpha, P^\beta_\alpha] = 0$.

In the following we will use local coordinates $(U; x^1, \dots, x^{2n})$ on $M$ which are adapted to the almost product structure $[D_1, D_2]$. This means that the holonomic frame $(\partial_i \mid \partial_{n+i})$ satisfies at any point $x \in U$:

$$(\partial_1, \dots, \partial_n)_x \text{ is a basis of } D_{1x}, \quad (\partial_{n+1}, \dots, \partial_{2n})_x \text{ is a basis of } D_{2x}.$$

We use the notation $\partial_i = \partial/\partial x^i$, $\partial_{n+i} = \partial/\partial x^{n+i}$, and similarly for derivatives of functions, $\dfrac{\partial f^i}{\partial x^j} = f^i_{x^j}$ etc.

**2. Projectors.** Let $(D_1, D_2, D_3)$ be a 3-web on $M$. About any point $p \in M$, choose a fixed local adapted chart $(U; x^i)$, $U \ni p$. In adapted coordinates, the

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projectors associated with the web are represented by $(2n, 2n)$-matrices as follows:

$$\text{(1)} \qquad \begin{aligned} P_1^2 &= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} & P_2^1 &= \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \\ P_1^3 &= \begin{pmatrix} I & 0 \\ -P & 0 \end{pmatrix} & P_3^1 &= \begin{pmatrix} 0 & 0 \\ P & I \end{pmatrix} \end{aligned}$$

where $P = (p_i^k)$, $p_i^k = p_i^k(x^1, \dots, x^{2n})$ is a regular $(n, n)$-matrix at any point, $I$ and $0$ denote zero and unit $(n, n)$-matrices respectively. It can be verified that

$$P_2^3 = \mathrm{id}_{D_2}, \qquad P_2^3 \mid D_1 = (P_1^3 \mid D_2)^{-1}.$$

Denoting by $\tilde{P} = (q_j^k)$ the inverse matrix of $P = (p_i^k)$, $p_i^k \cdot q_k^j = \delta_i^j$, we can write

$$P_3^2 = \begin{pmatrix} I & \tilde{P} \\ 0 & 0 \end{pmatrix}, \qquad P_2^3 = \begin{pmatrix} 0 & -\tilde{P} \\ 0 & I \end{pmatrix}.$$

The coordinate expressions of the web-associated fields $B_\gamma$ in adapted coordinates are

$$B_1 = P_1^2 - P_3^1 = \begin{pmatrix} I & 0 \\ -P & I \end{pmatrix}, \quad B_2 = P_2^3 - P_1^2 = \begin{pmatrix} -I & -\tilde{P} \\ 0 & I \end{pmatrix},$$

$$B_3 = P_3^1 - P_2^3 = \begin{pmatrix} 0 & \tilde{P} \\ P & 0 \end{pmatrix}.$$

Further,

$$P_1^2 B_3 = P_3^1 - P_2^1 = \begin{pmatrix} 0 & 0 \\ P & 0 \end{pmatrix}, \quad B_3 P_1^2 = P_3^2 - P_1^2 = \begin{pmatrix} 0 & \tilde{P} \\ 0 & 0 \end{pmatrix}.$$

**3. Regular webs.** A 3-web is *regular*[1] if it is locally equivalent with a 3-web formed by tangent bundles to three foliations of parallel $n$-planes (in general position) of an affine space $A_{2n}$. Locally, it can be realized as a Cartesian product of a commutative Lie group and its opposite group, $(L, \circ) \times (L, \circ^{op})$ where $x \circ^{op} y = y \circ x$.

Let us consider a change of local adapted coordinates

$$\Phi : \quad \begin{aligned} y^i &= f^i(x^1, \dots, x^n), & f^i_{x^{n+j}} &= 0, \\ y^{n+i} &= f^{n+i}(x^{n+1}, \dots, x^{2n}), & f^{n+i}_{x^j} &= 0. \end{aligned}$$

Now a 3-web is regular if and only if there is a local $[D_1, D_2]$-adapted coordinate system $(y^i)$ such that $D_3$ is a span of $\left\{ \dfrac{\partial}{\partial y^i} + \dfrac{\partial}{\partial y^{n+i}} \right\}_{i=1,\dots,n}$. The vector fields $(\frac{\partial}{\partial y^i} + \frac{\partial}{\partial y^{n+i}})$ constitue a basis of $D_3$ if and only if the system of equations $f_{n+i}^{n+j} - p_i^s f^j_{x^s} = 0$, $i, j = 1, \dots, n$ is satisfied where $p_i^s$ are components of the projector $P_3^1$ introduced in (1), [10].

[1] More frequently, *parallelizable*.

**Lemma 1.** *In the above notation, the system of partial differential equations*

$$f^i_{x^{n+j}} = 0, \quad f^{n+i}_{x^j} = 0, \quad f^{n+j}_{n+i} - p^s_i f^j_{x^s} = 0, \quad i, j, s = 1, \dots, n$$

*is solvable if and only if the following conditions are satisfied* [10, 12] :

$$(P_{x^{n+i}})^s_j = (P_{x^{n+j}})^s_i, \quad (\tilde{P}_{x^i})^s_j = (\tilde{P}_{x^j})^s_i, \tag{2}$$

$$P_{x^{n+j}x^s} = P_{x^{n+j}} \cdot \tilde{P} \cdot P_{x^s} = -P_{x^{n+j}} \cdot \tilde{P}_{x^s} \cdot P. \tag{3}$$

The condition (3) can be written as $(P_{x^{n+j}} \cdot \tilde{P})_{x^s} = 0$. In components, (2) takes the form

I. integrability condition: $\dfrac{\partial p^s_i}{\partial x^{n+j}} - \dfrac{\partial p^s_j}{\partial x^{n+i}} = 0,$

II. integrability condition: $\dfrac{\partial q^s_i}{\partial x^j} - \dfrac{\partial q^s_j}{\partial x^i} = 0.$

In the following, we will re-formulate (2) as a condition on Nijenhuis brackets of projectors.

**4. Evaluations in adapted coordinates.** Let us evaluate the Nijenhuis brackets of projectors on vector fields of the holonomic frame.[2]

$\{P^2_1, P^2_1\} = 0$ follows by integrability of $[D_1, D_2]$.

$$[P^3_1, P^2_1](\partial_i, \partial_j) = [P^3_1, P^2_1](\partial_i, \partial_{n+j}) = 0,$$

$$[P^3_1, P^2_1](\partial_{n+i}, \partial_{n+j}) = \left(\frac{\partial p^k_j}{\partial x^{n+i}} - \frac{\partial p^k_i}{\partial x^{n+j}}\right)\partial_k = ((P_{x^{n+i}})^k_j - (P_{x^{n+j}})^k_i)\,\partial_k;$$

$$[P^3_2, P^2_1](\partial_i, \partial_{n+j}) = [P^3_2, P^2_1](\partial_{n+i}, \partial_{n+j}) = 0,$$

$$[P^3_2, P^2_1](\partial_i, \partial_j) = \left(\frac{\partial q^k_i}{\partial x^j} - \frac{\partial q^k_j}{\partial x^i}\right)\partial_{n+k} = ((\tilde{P}_{x^j})^k_i - (\tilde{P}_{x^i})^k_j)\,\partial_{n+k}.$$

Further,

$$[P^2_1, B_3](\partial_i, \partial_{n+j}) = 0,$$
$$[P^2_1, B_3](\partial_i, \partial_j) = ((\tilde{P}_{x^j})^k_i - (\tilde{P}_{x^i})^k_j)\,\partial_{n+k},$$
$$[P^2_1, B_3](\partial_{n+i}, \partial_{n+j}) = ((P_{x^{n+i}})^k_j - (P_{x^{n+j}})^k_i)\,\partial_k.$$

So (2) is equivalent with a couple of conditions $[P^3_1, P^2_1] = [P^3_2, P^2_1] = 0$, or with a single condition $[P^2_1, B_3] = 0$. We will verify that if (2) holds then also the remaining couples of projectors vanish.

$$[P^3_1, P^3_2](\partial_i, \partial_j) = ((\tilde{P}_{x_j})^s_i - (\tilde{P}_{x_i})^s_j)(P)^k_s\partial_k + ((\tilde{P}_{x^j})^k_i - (\tilde{P}_{x^i})^k_j)\,\partial_{n+k} = 0,$$

$$\begin{aligned}[P^3_1, P^3_2](\partial_{n+i}, \partial_{n+j}) &= ((P_{x^{n+j}})^k_i - (P_{x^{n+i}})^k_j)\,\partial_k \\ &\quad + ((P_{x^{n+i}})^s_j - (P_{x^{n+j}})^s_i)(\tilde{P})^k_s\,\partial_{n+k} = 0,\end{aligned}$$

$$\begin{aligned}[P^3_1, P^3_2](\partial_i, \partial_{n+j}) &= ((\tilde{P}_{x^{n+j}})^s_i(P)^k_s + (\tilde{P})^s_i(P_{x^{n+s}})^k_j)\,\partial_k \\ &\quad - ((P)^s_j(\tilde{P}_{x^s})^k_i + (P_{x^i})^s_j(\tilde{P})^k_s)\,\partial_{n+k}.\end{aligned}$$

[2] We introduce $\{P, Q\} = \frac{1}{2}[P, Q]$ for commuting $P$ and $Q$.

By (2) and $(\tilde{P}\cdot P)^j_i=\delta^j_i$ we obtain

$$
\begin{aligned}
(\tilde{P}_{x^{n+j}})^s_i(P)^k_s+(\tilde{P})^s_i(P_{x^{n+s}})^k_j &= (\tilde{P}_{x^{n+i}})^s_j(P)^k_s+(\tilde{P})^s_i(P_{x^{n+j}})^k_s \\
&= \frac{\partial}{\partial x^{n+j}}(\tilde{P}\cdot P)^k_i=0.
\end{aligned}
$$

Similarly,

$$
(P)^s_j(\tilde{P}_{x^s})^k_i+(P_{x^i})^s_j(\tilde{P})^k_s=\frac{\partial}{\partial x^i}(\tilde{P}\cdot P)^k_j=0.
$$

Since web distributions are integrable the almost product structures $[D_1,D_3]$ and $[D_2,D_3]$ are also integrable, that is, $[P^3_1,P^3_1]=[P^3_2,P^3_2]=0$. This can be obtained also by the direct evaluation:

$$
\begin{aligned}
&\{P^3_1,P^3_1\}(\partial_i,\partial_j)=\{P^3_1,P^3_1\}(\partial_i,\partial_{n+j})=0,\\
&\{P^3_1,P^3_1\}(\partial_{n+i},\partial_{n+j})=((P)^s_i(P_{x^s})^k_j-(P)^s_j(P_{x^s})^k_i+(P_{x^{n+i}})^k_j-(P_{x^{n+j}})^k_i)\,\partial_k;\\
&\{P^3_2,P^3_2\}(\partial_i,\partial_{n+j})=\{P^3_2,P^3_2\}(\partial_{n+i},\partial_{n+j})=0,\\
&\{P^3_2,P^3_2\}(\partial_i,\partial_j)=((\tilde{P})^s_i(\tilde{P}_{x^{n+s}})^k_j-(\tilde{P})^s_j(\tilde{P}_{x^{n+s}})^k_i+(\tilde{P}_{x^i})^k_j-(\tilde{P}_{x^j})^k_i)\,\partial_{n+k}.
\end{aligned}
$$

By the second equality in (2), $(\tilde{P}_{x^t}P)^l_i=(\tilde{P}_{x^i}P)^l_t$ which we can rewrite in the form $(\tilde{P}P_{x^t})^l_i=(\tilde{P}P_{x^i})^l_t$. This is equivalent (using multiplicaton by $P$ twice) with the equality $(P)^t_k(P_{x^t})^l_m=(P)^i_m(P_{x^i})^l_k$. Similarly, by the first part of (2), $(P_{x^{n+i}}\tilde{P})^k_j=(P_{x^{n+j}}\tilde{P})^k_i$ holds. We can re-write this equality as $(P\tilde{P}_{x^{n+i}})^k_j=(P\tilde{P}_{x^{n+j}})^k_i$ which is equivalent with $(\tilde{P})^i_m(\tilde{P}_{x^{n+i}})^k_t=(\tilde{P})^j_t(\tilde{P}_{x^{n+j}})^k_m$. So we have proved:

**Proposition 1.** *The following conditions are equivalent:*

(i) $[P^2_1,B_3]=0$,
(ii) $[P^3_1,P^2_1]=[P^3_2,P^2_1]=0$,
(iii) $[P^\beta_\alpha,P^\delta_\gamma]=0\ \ (\alpha\neq\beta,\gamma\neq\delta)$.

**5. Chern connections.** A three-web manifold is endowed with three Chern connections $\overset{\gamma}{\nabla}$, $\gamma=1,2,3$

$$
\overset{\gamma}{\nabla}_X Y=\sum_{(\alpha,\beta)\in S_2}(P^\alpha_\gamma-P^\alpha_\beta)[P^\beta_\alpha X,P^\beta_\gamma Y]+P^\beta_\alpha[P^\alpha_\beta X,P^\beta_\alpha Y]
$$

such that

(i) $\overset{\gamma}{\nabla}_X P^\beta_\alpha Y=0$ for any vector fields $X$, $Y$ and $\alpha$, $\beta$, $\gamma\in\{1,2,3\}$ (all projectors of the web are covariantly constant),
(ii) for any permutation of indexis $(\alpha,\beta,\gamma)$, $\overset{\gamma}{T}(P^\beta_\alpha X,P^\alpha_\beta Y)=0$ (all couples of "homogeneous" vectors are conjugated with respect to the torsion tensor).

The torsion is given by the formula

$$\overset{\gamma}{T} = \sum_{(\alpha,\beta)\in S_2} (P^\alpha_\gamma - P^\alpha_\beta)\Big([P^\beta_\alpha X, P^\beta_\gamma Y] - [P^\beta_\gamma X, P^\beta_\alpha Y]\Big) + [P^\alpha_\beta X, P^\beta_\alpha Y] + [P^\beta_\alpha X, P^\alpha_\beta Y] - [X,Y],$$

and satisfies for $(\alpha,\beta,\gamma)\in S_3$:

$$\overset{\gamma}{T}(P^\beta_\alpha X, P^\alpha_\beta Y) = 0, \qquad P^\beta_\alpha \overset{\gamma}{T}(P^\alpha_\beta X, P^\alpha_\beta Y) = 0,$$
$$B_\gamma \overset{\gamma}{T}(P^\beta_\alpha X, P^\beta_\alpha Y) = -P^\alpha_\beta B_\gamma\{B_\gamma, B_\gamma\}(P^\beta_\alpha X, P^\beta_\alpha Y) = \overset{\gamma}{T}(B_\gamma P^\beta_\alpha X, B_\gamma P^\beta_\alpha Y).$$

On $D_\alpha$, the torsion $\overset{\gamma}{T}$ coincides with $B_\gamma[P^\beta_\alpha, B_\gamma]$, and on $D_\beta$, we obtain $\overset{\gamma}{T} = -B_\gamma[P^\beta_\alpha, B_\gamma]$. All web distributions are parallel with respect to the Chern connections, any distribution is autoparallel and parallel relative to any other. Any Chern connection is reducible to any leaf of the integral foliations of web distributions [10, 7]. With respect to $[D_\alpha, D_\beta]$-adapted coordinates, the only non-zero Christoffels of the Chern connection $\overset{\gamma}{\Gamma}$ are

$$\overset{\gamma}{\Gamma}{}^s_{ij} = \frac{\partial q^k_j}{\partial x^i}\, p^s_k, \qquad \overset{\gamma}{\Gamma}{}^{n+s}_{n+i,n+j} = \frac{\partial p^k_j}{\partial x^{n+i}}\, q^s_k$$

where $P = (p^j_i)$ is the $(n,n)$-matrix from the representation of the projector $P^\alpha_\gamma$ in $[D_\alpha, D_\beta]$- adapted coordinates, $P^\alpha_\gamma = \begin{pmatrix} 0 & 0 \\ P & I \end{pmatrix}$, $\tilde{P} = (q^j_i)$ denotes its inverse. Therefore the only non-zero components of the torsion tensor are

$$\overset{\gamma}{T}{}^s_{ij} = \left(\frac{\partial q^k_j}{\partial x^i} - \frac{\partial q^k_i}{\partial x^j}\right) p^s_k, \qquad \overset{\gamma}{T}{}^{n+s}_{n+i,n+j} = \left(\frac{\partial p^k_j}{\partial x^{n+i}} - \frac{\partial p^k_i}{\partial x^{n+j}}\right) q^s_k.$$

The curvature tensor satisfies

$$\overset{\gamma}{R} P^\beta_\alpha = P^\beta_\alpha \overset{\gamma}{R}, \qquad B_\gamma \overset{\gamma}{R} = \overset{\gamma}{R} B_\gamma,$$
$$\overset{\gamma}{R}(P^\beta_\alpha X, P^\beta_\alpha Y) = 0 \quad \text{for any vector fields } X,\ Y,$$

$$\begin{aligned}
&\overset{\gamma}{R}(P^\beta_\alpha X, P^\alpha_\beta Y) P^\beta_\alpha Z = \\
&- P^\beta_\alpha[P^\alpha_\beta Y, P^\beta_\alpha B_\gamma[P^\beta_\alpha X, B_\gamma P^\beta_\alpha Z]] - P^\beta_\alpha[P^\alpha_\beta[P^\beta_\alpha X, P^\alpha_\beta Y], P^\beta_\alpha Z] \\
&- P^\beta_\alpha B_\gamma[P^\beta_\alpha[P^\beta_\alpha X, P^\alpha_\beta Y], B_\gamma P^\beta_\alpha Z] + P^\beta_\alpha B_\gamma[P^\beta_\alpha X, B_\gamma P^\beta_\alpha[P^\alpha_\beta Y, P^\beta_\alpha Z]], \\
&\overset{\gamma}{R}(P^\beta_\alpha X, P^\alpha_\beta Y) P^\alpha_\beta Z = \\
&P^\alpha_\beta[P^\beta_\alpha X, P^\alpha_\beta B_\gamma[P^\alpha_\beta Y, B_\gamma P^\alpha_\beta Z]] - P^\alpha_\beta[P^\beta_\alpha[P^\beta_\alpha X, P^\alpha_\beta Y], P^\alpha_\beta Z] \\
&- P^\alpha_\beta B_\gamma[P^\alpha_\beta[P^\beta_\alpha X, P^\alpha_\beta Y], B_\gamma P^\alpha_\beta Z] - P^\alpha_\beta B_\gamma[P^\alpha_\beta Y, B_\gamma P^\alpha_\beta[P^\beta_\alpha X, P^\alpha_\beta Z]].
\end{aligned}$$

If we evaluate the components of $\overset{\gamma}{R}$ in $[D_\alpha, D_\beta]$-adapted coordinates we obtain

$$\overset{\gamma}{R}(\partial_i, \partial_{n+j})\partial_k = -P^\beta_\alpha[\partial_{n+j}, (P^\alpha_\gamma - P^\alpha_\beta)[\partial_i, (P^\beta_\gamma - P^\beta_\alpha)\partial_k]],$$
$$\overset{\gamma}{R}(\partial_i, \partial_{n+j})\partial_{n+k}[\partial_i, (P^\beta_\gamma - P^\beta_\alpha)[\partial_{n+j}, (P^\alpha_\gamma - P^\alpha_\beta)\partial_{n+k}]].$$

It follows that the non-zero components of the curvature tensor are

$$\overset{\gamma}{R}{}^{n+s}_{i,n+j,n+k} = (C(x^i, x^{n+j}))^s_k \qquad \overset{\gamma}{R}{}^s_{i,n+j,k} = (\tilde{P} \cdot C(x^i, x^{n+j}) \cdot P)^s_k$$

where the matrices $C$ are given by $C(x^i, x^{n+j}) = (P_{x^{n+j}}\tilde{P})_{x^i}$, $P = P^\beta_\alpha$. Therefore the condition $\overset{\gamma}{R} = 0$ is equivalent with (3).

The curvature and torsion tensors play a central role by classification of webs. Especially, a 3-web is regular if and only if $\overset{\gamma}{T} = \overset{\gamma}{R} = 0$ [1, 3]. The explicit formula for evaluation of $\overset{\gamma}{R}$ through projectors can be also established but is complicated.

**6. Paratactical webs.** A 3-web for which $\overset{\gamma}{T}$ vanish is called *paratactical* [1]. Paratactical webs are characterized through projectors as follows.

**Proposition 2.** *The following conditions are equivalent for any fixed permutation* $(\alpha, \beta, \gamma)$ *of indexis:*

(i) $\overset{\gamma}{T} = 0$,
(ii) $[P^\beta_\alpha, B_\gamma] = 0$,
(iii) $[P^\gamma_\alpha, P^\beta_\alpha] = [P^\gamma_\beta, P^\alpha_\beta] = 0$,
(iv) *the Nijenhuis brackets vanish for all couples of projectors.*

**Theorem.** *A 3-web is paratactical if and only if Nijenhuis tensors vanish for all couples of projectors.*

As a consequence we obtain:

**Proposition 3.** *If one of the Chern connections is symmetric (=torsion-free) so are the other.*

## References

[1] M. A. Akivis, *Three-webs of multidimensional surfaces*, Trudy geom. seminara **2** (1969), 7–31. (Russian)

[2] M. A. Akivis, A. M. Shelekhov, *Contravariant theory of multidimensional three-webs*, Webs & Quasigroups (1993), 20–32, Tvers' State University, Tversk, Russia.

[3] S. S. Chern, *Eine Invariantentheorie der 3-Gewebe aus r-dimensionalen Mannigfaltigkeiten in* $R^{2n}$, Abhandl. Math. Sem. Hamburg Univ. **11** (1936), 333–358.

[4] V. V. Goldberg, *Theory of multicodimensional* $(n+1)$*-webs*, Kluwer Academic Publishers, Dordrecht (Boston, London), 1988.

[5] M. Kikkawa, *Canonical connection of homogeneous Lie loops and 3-webs*, Mem. Shimane Univ., Nat. Sci. **19** (1985).

[6] P. T. Nagy, *On the canonical connection of a three-web*, Publ. Math. Debrecen **32** (1985), 93-99.

[7] P. T. Nagy, *Invariant tensor fields and the canonical connection of a 3-web*, Aeq. Math. **35** (1988), 31-44, University of Waterloo, Birkhäuser Verlag, Basel.

[8] A.M. Shelekhov, *A good formula for 3-web curvature tensor*, Webs & Quasigroups (1993), 44-50, Tvers' State University, Tversk, Russia.

[9] A. Vanžurová, *On $(3, 2, n)$-webs*, Acta Sci. Math. **59** (1994), 657–677, Szeged.

[10] A. Vanžurová, *Parallelisability conditions for three-web manifolds*, Arch. Math. **31** (1995), 74–84, Brno.

[11] A. Vanžurová, *On torsion of a three-web*, submitted.

[12] Th. M. Rassias , J. Simsa, *Finite Sums Decompositions in Mathematical Analysis*, John Wiley and Sons, Chichester, New York, Brisbane, Toronto, Singapore, 1995.

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