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GRAPHS, EHRESMANN CONNECTIONS AND VANISHING CYCLES

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In recent years the graph of a foliation, an object which has been known for a long time, cf. [10], has known new interest. In fact there are two groupoids associated with a foliation, the homotopy groupoid and the holonomy groupoid, sometimes called the graph. It serves as a basis for the construction of the C^*-algebra associated to the foliation. Moreover, the homotopy groupoid of the characteristic foliation of a Poisson manifold is used in the symplectic integration of this Poisson manifold, cf. [22, 12, 15, 2]. For a general foliation we do not know whether its groupoids are Hausdorff manifolds, cf. [23, 21]. Recently P. Dazord and G. Hector proved that the homotopy groupoid is Hausdorff iff the foliation has no vanishing cycles, cf. [12]. In this note study the existence of vanishing cycles for some classes of foliations, including totally geodesic foliations and these which admit an Ehresmann connection.

1. Preliminaries

In this section, for the convenience of the reader, we will recall some basic definitions and results.

1.1. The graph of a foliation

The definition and the basic properties of the graph of a foliation can be found in [23]. Let us recall the definition.

The graph $GR(\mathcal{F})$ of the foliation $\mathcal{F}$ is the space of equivalence classes of triples (y, α, x) where x and y are points of the same leaf L of $\mathcal{F}$ and α is a path in L linking x to y. Two triples (y, α, x) and (y', α', x') are equivalent iff $x = x'$, $y = y'$ and the holonomy of the curve $\alpha^{-1} * \alpha'$ is trivial. In the same paper the author

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proved that the graph of a foliation is a manifold of dimension $n+p$ but in general non–Hausdorff. Moreover, if the elements of the holonomy pseudogroup are determined by their jets the graph is a Hausdorff topological space.

The correspondences

$$p_1\colon \langle y,\alpha,x\rangle \longmapsto x,$$

the source projection, and

$$p_2\colon \langle y,\alpha,x\rangle \longmapsto y,$$

the target projection, define two submersions

$$p_1\colon GR(\mathcal{F}) \longrightarrow M \quad \text{and} \quad p_2\colon GR(\mathcal{F}) \longrightarrow M.$$

First Winkelnkemper and then Wolak, cf. [23, 25, 28], studied the local triviality of the submersions p_1 and p_2. We know that under natural assumptions they are local trivial fibre bundles for Riemannian, transversely affine, totally geodesic and some other classes of foliations. Using the method developed in [25, 28] one can prove that these submersions are locally trivial fibre bundles for foliations admitting an Ehresmann connection.

The homotopy groupoid of a foliation is defined in a similar way, we identify leaf curves which are homotopic relative to their ends in the corresponding leaf. Likewise the homotopy groupoid is a manifold, but not necessarily Hausdorff. There is a natural submersion from the homotopy groupoid onto the holonomy groupoid of a given foliation. In [13] one can find an example of a foliation for which the holonomy groupoid is Hausdorff but its homotopy groupoid is not, cf. also [17]. The non-Hausdorffness of the homotopy groupoid is an important obstruction in the symplectic integration of Poisson manifolds, cf. [13, 2], as well as in the quantization, cf. [1].

1.2. Ehresmann connections

The concept of an Ehresmann connection appears for the first time, under a different name, in [10]. However, recent studies of totally geodesic foliations have brought out its importance, cf. [6, 9], in geometry.

The notion of a foliation admitting an Ehresmann connection is a natural generalization of the concept of a totally geodesic foliation, cf. [7], namely:

Let $(M,\mathcal{F})$ be a foliated manifold. A supplementary subbundle Q to $T\mathcal{F}$ is called an Ehresmann connection if for any point $x\in M$ and any pair of curves $\alpha\colon[0,a]\to M$ and $\beta\colon[0,b]\to M$ with the same starting point, such that the curve α is a leaf curve, and β is tangent to Q, there exists a smooth mapping $\sigma\colon[0,a]\times[0,b]\to M$ such that

1) the curves $\sigma_s\colon[0,a]\to M$, $\sigma_s=\sigma|[0,a]\times\{s\}$, $s\in[0,b]$, are leaf curves;

2) $\sigma_0=\alpha$;

3) the curves $\sigma^t\colon[0,b]\to M$, $\sigma^t=\sigma|\{t\}\times[0,b]$, $t\in[0,a]$, are tangent to Q;

4) $\sigma^0=\beta$.

If rectangles as above exist only for a certain class of curves tangent to the subbundle Q we say that Q is a reduced Ehresmann connection. This class must be sufficiently rich to contain curves starting in each direction of Q. For applications it could be usefull to assume an equivalent definition allowing the path β to begin at any point of the curve α, i.e. $\beta(0) = \alpha(t_0)$ for some $t_0 \in [0, a]$. Then we would have the condition 4') $\sigma^{t_0} = \beta$.

We know that the orthogonal bundle to a totally geodesic foliation on a complete Riemannian manifold is an Ehresmann connection for this foliation, cf. [6].

Very many results concerning Riemannian foliations were proved using the properties of geodesics under the assumption that the Riemannian manifold is complete. However, in most cases it is sufficient to know that the orthogonal subbundle is an Ehresmann connection and to demonstrate that it is an Ehresmann connection it is not necessary to assume that the Riemannian metric is complete. Let us give an example.

Example Let $M = R^3 - \{(0,0)\} \times R$ and $p\colon M \to R \times R$ be the projection $(x, y, z) \mapsto (x, y)$. Then $p(M) = R^2 - \{(0, 0)\}$. With the standard metric g_0 on M induced from the euclidean space R^3, the Riemannian manifold (M, g_0) is not complete; however, the orthogonal bundle to the foliation defined by the connected components of the fibres of the submersion p is an Ehresmann connection for this foliation. To put it succinctly, we can define and work with an Ehresmann connection when certain geodesics are defined *uniformly* along the leaves.

The existence of the smooth mapping (rectangle) σ allows us to push any Q-curve β along the curve α and any leaf curve α along the curve β. Namely, cf. [6],

$$\beta_\sharp : \alpha \mapsto \beta_\sharp \alpha = \sigma_b$$

and

$$\alpha_\sharp : \beta \mapsto \alpha_\sharp \beta = \sigma^a$$

Let $K(L, x)$ be the subgroup of $\pi_1(L, x)$ of the homotopy classes of loops τ in L for which $\tau_\sharp \gamma = \gamma$ for any curve γ tangent to Q and starting at the point x. In [6], the authors notice that $K(L, x)$ is a normal subgroup of $\pi_1(L, x)$. The quotient group $\pi_1(L, x)/K(L, x)$ we denote by $H_g(L, x)$, cf. [7, 9].

At the end of this section let us formulate some basic properties of an Ehresmann connection.

Lemma 1. For any rectangle σ defined by a pair of curves (α, β), the curve σ^a depends only on the homotopy class of the curve α and the curve σ_b on the homotopy class of β.

The above lemma is a consequence of the following one:

Lemma 2. Let $\gamma\colon [0, c] \times [0, a] \to M$ be a smooth (continuous) family of leaf curves parametrized by the interval $[0, c]$. Let $\beta\colon [0, c] \times [0, b] \to M$ be a smooth, resp. continuous, family of curves tangent to the subbundle Q such that $\gamma|[0, c] \times \{0\} = \beta|[0, c] \times \{0\}$. Then there exists a smooth, resp. continuous, mapping $\sigma\colon [0, c] \times [0, a] \times [0, b] \to M$ such that for any $t \in [0, c]$ the mapping $\sigma|\{t\} \times [0, a] \times [0, b]$ is the rectangle for the pair (α^t, β^t).

1.3. Some examples

For some classes of foliations one can build the rectangle for some pairs of curves, eg. for a transversely affine foliation it is possible for any leaf curve and the other curve being a transverse geodesic of the transversely projectable flat connection, cf. [27, 26], provided that the foliation is transversely geodesically complete. The same is true for any foliation admitting a tranversely complete tranversely projectable connection, cf. [26]. More generally it is also true for foliations admitting foliated systems of ordinary differential equations (FSODE) which are transversely complete, cf. [24, 26]. Moreover, a transverse curve instead of being a geodesic (solution of an FSODE) can be a piecewise geodesic (resp. piecewise solution of an FSODE). A similar argument is valid for foliations parallel with respect to some connection. It is the case of a lagrangian foliation, cf. [18, 15, 16, 29].

2. Vanishing cycles

A vanishing cycle for a foliation $\mathcal{F}$ it is a mapping $c\colon S^1\times[0,1]\to M$ such that for any $t\in[0,1]$ $c_t=c|S^1\times\{t\}$ is a loop on a leaf of $\mathcal{F}$ and c_0 is not homotopic to the constant loop in the leaf but for $t>0$ the loops c_t are, cf. [14].

In [12] it is demonstrated that the non-existence of vanishing cycles is equivalent to the Hausdorffness of the homotopy groupoid of the foliation, see also [2]. Using the same method one can prove the following theorem. Before stating the theorem we have to introduce the notion of a holonomy vanishing cycle.

Definition 1. A holonomy vanishing cycle for a foliation $\mathcal{F}$ it is a mapping $c\colon S^1\times[0,1]\to M$ such that for any $t\in[0,1]$ $c_t=c|S^1\times\{t\}$ is a loop on a leaf of $\mathcal{F}$ and c_0 has non-trivial holonomy but for $t>0$ the loops c_t carry no holonomy.

Theorem 1. Let $(M,\mathcal{F})$ be a foliated manifold. Then the graph of the foliation $\mathcal{F}$ is Hausdorff iff $\mathcal{F}$ has no holonomy vanishing cycles.

In particular we have the following:

Corollary 1. Let $(M,\mathcal{F})$ be a foliated manifold. If $\mathcal{F}$ has no holonomy, then the graph of the foliation $\mathcal{F}$ is a Hausdorff manifold.

For foliations admitting a (reduced) Ehresmann connection we have an interesting characterization of the existence of (holonomy) vanishing cycles.

We are going to prove that for the foliations we are interested in the existence of a vanishing cycle implies the existence of some very special vanishing cycles.

Lemma 3. If a foliation $\mathcal{F}$ admits a vanishing cycle, then it admits a vanishing cycle $c\colon S^1\times[0,1]\to M$ for which the curves c^t are tangent to the subbundle Q.

Proof Let us first notice that there exists a vanishing cycle, denoted by the same letter c, based on the same loop c_0 for which the curve c^0 is tangent ot the subbundle Q. Then we apply Lemma 2 to the mapping c with the second mapping given by $\beta\colon[0,1]\times[0,1]\to M$ $\beta(t,s)=\sigma^0(s+t(1-s))$. From the continuity assured by Lemma 2 we get that $c_0^\sharp(c^0)=c^0$. This fact ensures that the rectangle defined by the pair (c_0, c^0) defines a mapping σ from $S^1 \times [0,1]$ into M. We know that the loops σ_t and c_t are in the same leaf for any $t \in [0,1]$ and from the continuity we obtain that for t small enough these loops are so near that they are free homotopic. Therefore the loops $\sigma_t, t > 0$ are nullhomotpic and the mapping $\sigma|S^1 \times [0,t_0] \to M$ is a vanishing cycle for a suitable small $t_0 > 0$.□

Theorem 2. A foliation admitting an Ehresmann connection has no vanishing cycles.

Proof Let assume that there exists a vanishing cycle and take a one whose existence has been ensured by Lemma 3: $\sigma\colon S^1 \times [0,1] \to M$. For any $t \in (0,1]$ $\sigma_t\colon S^1 \to M$ is a leaf loop homotopic to the constant loop in the leaf L_t passing through the point $\sigma(0,t)$. Let $H\colon [0,1]\times[0,1] \to L_1$ be the homotopy of σ_1 to the constant loop. Let $\omega_t\colon [0,1]\times[0,1] \to M$ be the rectangle defined by the pair $(H_t, (\sigma^0)^{-1})$. For some $t \in [0,1]$ $\omega_t^0 \neq \omega_t^1$, as otherwise the loop σ_0 would be homotopic to the constant loop in the leaf L_0. Contradiction. □

Corollary 2. The homotopy groupoid of a foliation admitting an Ehresmann connection is a Hausdorff manifold.

The existence of holonomy vanishing cycles can be characterized in the language of holonomy groups of leaves.

Theorem 3. Let $\mathcal{F}$ be a foliation admitting a (reduced) Ehresmann connection. Then there exists a holonomy vanishing cycle iff the natural mapping $\xi\colon H_g(L) \to H_{tr}(L)$ is not injective for some leaf L of the foliation $\mathcal{F}$.

Proof $\to$) Let $\sigma\colon S^1 \times [0,1] \to M$ be a holonomy vanishing cycle. The loop σ_0 carries nontrivial holonomy. Then there exits a holonomy vanishing cycle based on the same loop σ_0 for which the curve σ^0 is tangent to the transverse subbundle Q. Let us denote this new holonomy vanishing cycle by the same letter σ. If in the rectangle c defined by the pair $(\sigma_1, (\sigma^0)^{-1})$ c^0 is different from c^1, the loop σ_1 and the constant loop in the leaf L_1 define different elements in the group $H_g(L_1)$ but both have trivial holonomy. The same is true for any loop σ_t and the curves $\sigma^0|[0,t]$ and $\sigma^0|[t,1]$. If not, then for any $t > 0$ $\sigma_t^\sharp(\sigma^0) = \sigma^0$, and by continuity $\sigma_0^\sharp(\sigma^0) = \sigma^0$. The loop σ_0 carries holonomy h, so let y be a point of a transverse submanifold at the point $\sigma(0,0)$ for which $h(y) \neq y$. Let t be sufficiently small so that there exists a curve β tangent to Q starting at $\sigma(0,t)$ and ending at the plaque containing y. As the holonomy along σ_0 maps y to $h(y)$ $\beta(1)$ must be different from $\sigma_t^\sharp(\beta)(1)$, so the loop σ_t carries no holonomy but defines a nontrivial element in the group $H_g(L_t)$.

$\leftarrow$) Let $\sigma_0\colon [0,1] \to L$ be a loop in the leaf L which defines a nontrivial element in $H_g(L)$ but which carries no holonomy. Thus there exists a (distinguished) transverse curve β such that in the rectangle σ defined by the pair (σ_0, β) the transverse curve σ^0 is different from σ^1. However as the holonomy of σ_0 is trivial there exists $\epsilon_0 > 0$ such that $\sigma^0|[0,\epsilon_0] = \sigma^1|[0,\epsilon_0]$ and it is not the case for any $\epsilon_1 > \epsilon_0$, i.e. $\sigma^0|[\epsilon_0, t] \neq \sigma^1|[\epsilon_0,t]$ for any $t \in (\epsilon_0, \epsilon_1]$. Therefore the holonomy along the loop σ_{ϵ_0} is not trivial. The space of $t \in [0,\epsilon_0]$ for which the loop σ_t carries no holonomy is an open subset of $[0, \epsilon]$. Therefore there exists an $\epsilon > 0$ such that for any $t \in [0, \epsilon)$ the loop σ_t has no holonomy but the holonomy of σ_ϵ is nontrivial. Thus $\bar{\sigma}(t, s) = \sigma(t, \epsilon - t)$ for $(t, s) \in S^1 \times [0, \epsilon]$ is a holonomy vanishing cycle. □

Corollary 3. Let $(M, \mathcal{F})$ be a foliated manifold. If $\mathcal{F}$ admits a (reduced) Ehresmann connection, then the holonomy groupoid (graph) of $\mathcal{F}$ is Hausdorff iff for any leaf L of the foliation $\mathcal{F}$ the mapping $\xi\colon H_g(L) \to H_{tr}(L)$ is injective.

Having proved the characterization of the existence of (holonomy) vanishing cycles it is time to check in which cases these conditions are satisfied.

In some papers we studied, cf. [24, 25, 26], foliations admitting foliated systems of ordinary differential equations (FSODE). This class of foliations includes Riemannian, cf. [20], transversely affine, cf. [11, 27, 26], $\nabla - G$–, cf. [19, 20, 26], transversely projective, transversely conformal, cf. [4], foliations as well as foliations admitting a transversely projectable Cartan connection, cf. [3]. One of the main features of such foliations is that there exists a locally trivial fibre bundle $E_0 \to M$ over M with a foliation $\mathcal{F}_E$ whose leaves are covering spaces of the corresponding leaves of $\mathcal{F}$, [24, 26]. This bundle is the bundle of transverse initial conditions for the FSODE. In the cases we are interested in the leaves of the foliation $\mathcal{F}_E$ are holonomy coverings of leaves of the foliation $\mathcal{F}$. If transverse solutions exists globally, the foliation admits a reduced Ehresmann connection, cf. [24, 26]. Namely, let α be any leaf curve and β a transverse solution tangent to the chosen transverse subbundle Q starting at the point $x = \beta(0) = \alpha(0)$. Let $v \in (E_0)_x$ be the initial condition of the solution β. Let L_v be the leaf of the foliation $\mathcal{F}_E$ passing through the point v. As the leaf L_v is a (holonomy) covering space of the leaf L_x passing through the point x the leaf curve α can be lifted to a leaf curve $\bar{\alpha}$ in the leaf L_v starting at the point v. The curve $\bar{\alpha}$ provides us with initial conditions along the curve α. The rectangle is defined as follows: $\sigma(t, s) = exp(s; \bar{\alpha}(t))$ where $s \mapsto exp(s; \eta)$ is the solution of the FSODE with the initial condition η. Let α_1 and α_2 be two leaf curves from the point x to the point y and β a solution starting at the point x with the initial condition v. If our curves carry the same holonomy, their lifts $\bar{\alpha}_i$, $i = 1, 2$, respectively, to the point v of the bundle of initial conditions have the same end point, i.e. $\bar{\alpha}_1(1) = \bar{\alpha}_2(1)$. Let σ_1 and σ_2 be the rectangles associated to the pairs (α_1, β) and (α_2, β), respectively, then $(\sigma_1)^1 = (\sigma_2)^1$ as both curve are the solution of the FSODE with the initial condition $\bar{\alpha}_1(1) = \bar{\alpha}_2(1)$. Therefore we have proved the following proposition.

Proposition 1. Let $(M, \mathcal{F})$ be a foliated manifold admitting a transversely complete, transversely transitive FSODE with the uniqueness solution property. Then the foliation admits neither holonomy vanishing cycles nor vanishing cycles.

As a corollary of Proposition 1 and Theorem 1 we get the following theorem, cf. [25].

Theorem 4. Let $(M, \mathcal{F})$ be a foliated manifold admitting a transversely complete, transversely transitive FSODE with the uniqueness solution property. Then the homotopy groupoid and the holonomy groupoid (the graph) of the foliation $\mathcal{F}$ are Hausdorff manifolds.

As particular cases of this theorem we obtain the fact that under suitable assumptions the homotopy and holonomy groupoids of Riemannian, transversely affine and so on, foliations are Hausdorff manifolds. Therefore we have the following:

Theorem 5. Let $\mathcal{F}$ be a transversely complete Riemannian (transversely affine, $\nabla - G$) foliation, then the foliation admits neither holonomy vanishing cycles nor vanishing cycles.

For codimension 1 foliatons we can draw the following conclusion:

Corollary 4. Let $\mathcal{F}$ be a codimension one transversely complete transversely affine $(\nabla - G)$ foliation of a compact 3-manifold M, then the foliation does not have any Reeb component.

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