DIFFERENTIAL GEOMETRY AND APPLICATIONS
Proc. Conf., Aug. 28 – Sept. 1, 1995, Brno, Czech Republic
Masaryk University, Brno 1996, 353–372

ON THE VARIATIONAL CHARACTERIZATION OF GENERALIZED JACOBI EQUATIONS

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Abstract. We study higher–order variational derivatives of a generic second–order Lagrangian $\mathcal{L}_0 = \mathcal{L}_0(x, \phi, \partial\phi, \partial^2\phi)$ and in this context we discuss the Jacobi equation ensuing from the second variation of the action. We exhibit the different integrations by parts which may be performed to obtain the Jacobi equation and we show that there is a particular integration by parts which is invariant. We introduce two new Lagrangians, $\mathcal{L}_1$ and $\mathcal{L}_2$, associated to the first and second–order deformations of the original Lagrangian $\mathcal{L}_0$ respectively; they are in fact the first elements of a whole hierarchy of Lagrangians derived from $\mathcal{L}_0$. In terms of these Lagrangians, we are able to establish simple relations between the variational derivatives of different orders of a given Lagrangian. We then show that the Jacobi equations of $\mathcal{L}_0$ may be obtained as variational equations, so that the Euler–Lagrange and the Jacobi equations are obtained from a single variational principle based on the first–order variation $\mathcal{L}_1$ of the Lagrangian. We can furthermore introduce an associated energy–momentum tensor $\mathcal{H}^\mu{}_\nu$, which turns out to be a conserved quantity if $\mathcal{L}_0$ is independent of space–time variables.

0. Introduction

As is well known, the second variation of an action functional governs the behaviour of the action itself in the neighbourhood of critical sections. In particular, the Hessian of the Lagrangian defines a quadratic form whose sign properties allow to distinguish between minima, maxima and degenerate critical sections (see, *e.g.*, [1]). It is also well known that in the case of geodesics in a Riemannian manifold those fields which govern the transition from geodesics to geodesics, (*i.e.* those vectorfields which make the second variation to vanish identically modulo boundary terms) are called *Jacobi fields* [2] and they are solutions of a second–order differential equation known as *Jacobi equation* (of geodesics).

The notion of Jacobi equation as an outcome of the second variation is in fact fairly more general than this and general formulae for the second variation and

1991 *Mathematics Subject Classification*. 70 H 20, 70 H 30.
Key words and phrases. Calculus of variations, Jacobi equations.
This paper is in final form and no version of it will be submitted for publication elsewhere.

generalised Jacobi equations along critical sections have been already considered in the Calculus of Variations from a "structural viewpoint" (see, *e.g.*, the review of results contained in [3]). To our knowledge, however, in most of the current literature on the subject the second variation of functionals has been considered in a direct way and without resorting to general expressions, while integration by parts to reduce it to more suitable forms have been usually performed by *ad hoc* procedures, in spite of the fact that fairly general such formulae can be easily worked out (see below for a comment on a paper by Taub [4], which is appropriate to mention here but is better to discuss later). Because of these facts, we have reached the conclusion that the theory of second variations is worthy of being revisited, also in view of a number of applications which shall be mentioned later and will form the subject of forthcoming investigations.

In a previous paper [5], working in the general and global framework of first–order variational principles in fibered manifolds and their jet–prolongations, the notion of *generalised Jacobi equation* was developed and discussed for first–order Lagrangians. As is well known, the classical Jacobi equation for geodesics of a Riemannian manifold defines in fact the Riemann curvature tensor of g; because of this we can say that the second variation $\delta^2\mathcal{S}$ and the generalised Jacobi equations define the "curvature" of any given variational principle. In the generic case, of course, this notion has very little to say; in a continuation [6] of that work it was however shown that this general concept of "curvature" takes a particularly significant form in the case of generalised harmonic Lagrangians, giving rise to suitable "curvature tensors" which satisfy suitable "generalised Bianchi identities". Applications to second variations of relativistic Lagrangians (*i.e.* Lagrangians depending on the full curvature of a Riemannian metric) are being considered in [7] and [8].

For the sake of completeness, we should mention that a number of recent and less recent papers (see, *e.g.*, [9],[10] and ref.s quoted therein) have attacked the problem of "generalised Jacobi equations" and "curvature" for arbitrary second–order differential equations, from the general viewpoint of dynamical systems on a tangent bundle. These interesting contributions have a rather different nature from ours, since they refer to fairly general dynamical structures while the results we are going to discuss steam directly from the richer structure of Lagrangian systems on generic fiber bundles. Nevertheless, it would be interesting to compare the two viewpoints, also to check how much of our direct, simpler and straightforward results might be recovered from a suitable application of the more complicated structures discussed in [10] (and ref.s quoted therein) in a different and complementary framework. We hope to address this problem in future investigations.

In this paper we shall consider some further results related to the generalised Jacobi equation. For the sake of simplicity and for mere notational convenience we will establish our results only for second–order Lagrangians in field theory, although an analogous scheme can be easily worked out for higher–orders without bringing any new really new insight into the problem.

The first part of this paper (Section 1) is devoted to briefly recall the situation for first–order theory, as already discussed in [11] in the intrinsic language of differential geometry of jet–bundles. In Section 2 we consider the second variation

$\delta^2\mathcal{S}$ of an action $\mathcal{S}$, defined by a Lagrangian depending at most on second–order derivatives of the fields. We then show how a number of different integrations by parts allow to recast the Hessian in more suitable forms, which contain the Euler–Lagrange equations and define some ordinary differential equations of the second–order which we call the *generalised Jacobi equations*. These are in fact the equations which define along critical curves those vectorfields in the configuration space which make $\delta^2\mathcal{S}$ to vanish identically (modulo boundary conditions).

In Section 3 we introduce then a series of relevant relations between the Euler-Lagrange equation for a given Lagrangian $\mathcal{L}_0$, namely

$$\frac{\delta\mathcal{L}_0}{\delta\phi^A} \equiv \frac{\partial\mathcal{L}_0}{\partial\phi^A} - \frac{d}{dx^\mu}\left(\frac{\partial\mathcal{L}_0}{\partial{\phi^A}_\mu}\right) + \frac{d^2}{dx^\mu dx^\nu}\left(\frac{\partial\mathcal{L}_0}{\partial{\phi^A}_{\mu\nu}}\right) = 0 \tag{0.1}$$

and the "Jacobi equation" for the same Lagrangian $\mathcal{L}_0$, namely

$$\begin{aligned} J_A \equiv{}& \frac{\partial^2\mathcal{L}_0}{\partial\phi^A\partial\phi^B}\eta^B + \frac{\partial^2\mathcal{L}_0}{\partial\phi^A\partial{\phi^B}_\lambda}{\eta^B}_\lambda + \frac{\partial^2\mathcal{L}_0}{\partial\phi^A\partial{\phi^B}_{\lambda\rho}}{\eta^B}_{\lambda\rho} \\ &-\frac{d}{dx^\mu}\left(\frac{\partial^2\mathcal{L}_0}{\partial{\phi^A}_\mu\partial\phi^B}\eta^B + \frac{\partial^2\mathcal{L}_0}{\partial{\phi^A}_\mu\partial{\phi^B}_\lambda}{\eta^B}_\lambda + \frac{\partial^2\mathcal{L}_0}{\partial{\phi^A}_\mu\partial{\phi^B}_{\lambda\rho}}{\eta^B}_{\lambda\rho}\right) \\ &+\frac{d^2}{dx^\mu dx^\nu}\left(\frac{\partial^2\mathcal{L}_0}{\partial{\phi^A}_{\mu\nu}\partial\phi^B}\eta^B + \frac{\partial^2\mathcal{L}_0}{\partial{\phi^A}_{\mu\nu}\partial{\phi^B}_\lambda}{\eta^B}_\lambda + \frac{\partial^2\mathcal{L}_0}{\partial{\phi^A}_{\mu\nu}\partial{\phi^B}_{\lambda\rho}}{\eta^B}_{\lambda\rho}\right) = 0 \end{aligned} \tag{0.2}$$

Starting from the Lagrangian $\mathcal{L}_0$ we shall in fact define new Lagrangians $\mathcal{L}_1$ and $\mathcal{L}_2$ associated respectively to the first–order deformation of $\mathcal{L}_0$ and to the second–order deformation of $\mathcal{L}_0$, with the remarkable property that the Euler–Lagrange equations (0.1) and the Jacobi equations (0.2) of the original Lagrangian $\mathcal{L}_0$ can be rewritten together in terms of $\mathcal{L}_1$ and $\mathcal{L}_2$ as follows:

$$\frac{\delta\mathcal{L}_0}{\delta\phi^A} = \frac{\delta\mathcal{L}_1}{\delta\eta^A} = 0 \tag{0.3a}$$

$$\frac{\delta\mathcal{L}_2}{\delta\eta^A} = \frac{\delta\mathcal{L}_1}{\delta\phi^A} = 0 \tag{0.3b}$$

Therefore, we can consider the above relations as coming from a single variational principle based on the new Lagrangian $\mathcal{L}_1$ where, however, ϕ's and η's are considered as independent variables. This result finds important applications to Riemannian Geometry [11].

We can thence introduce momenta canonically conjugated to ϕ's and η's by means of standard prescriptions (see Section 3 below) as well as a canonical energy–momentum tensor ${\mathcal{H}^\mu}_\nu$, which is conserved if $\mathcal{L}_0$ does not depend explicitly on "space–time variables" x^λ. Therefore, the allowed deformations are selected by the first–order condition

$$\frac{d{\mathcal{H}^\mu}_\nu}{dx^\mu} = 0 \tag{0.4}$$

which replaces the second-order condition given by the Jacobi equation (0.2).

We are now in position to mention in a greater detail the paper by Taub [4] we already quoted above. In that paper, which is explicitly devoted to an application of second–order variations to relativistic fluid–dynamics, a general formula for the Jacobi equation is explicitly mentioned and it is claimed that a "well–known result" states that this equation is in fact the variational derivative of the Hessian $\mathcal{L}_2$ (see eq. (0.3b) above). Unfortunately, no reference is given to the source of this result; we stress, however, that the viewpoint we shall discuss in this paper is fairly different and more general. In fact, we shall not see Jacobi equations alone as (partial with respect to η) Lagrange equations of the Hessian $\mathcal{L}_2$ (as in [4]), but rather the system formed by Jacobi equations and the original Lagrange equations of $\mathcal{L}_0$ as variational equations of the first–order deformed Lagrangian $\mathcal{L}_1$.

Section 4 is finally devoted to some applications and examples.

For the sake of simplicity most of this paper will be written in the language of fundamental calculus in $\mathbf{R}^n$, although all the results presented here can be easily expressed and derived in the intrinsic language of fiber bundles.

1. The Lagrangian Characterization of Generalized Jacobi Equation in the First–Order Case

Let us first recall some basic notion from the Calculus of Variations on fibered manifolds. Let (B, M, π) be a fibered manifold and $\mathcal{L}_0 : J^1\pi \to \Lambda^n(T^*M)$ be a first–order Lagrangian (density) on B. Here $J^1\pi$ denotes the first–jet prolongation of π and $\Lambda^n(T^*M)$ is the bundle of n–forms of M, n being the dimension of the base manifold M. Locally:

$$\mathcal{L}_0 = L(x^\lambda,\, \phi^A,\, \phi^A_\lambda)\, ds \tag{1.1}$$

where $(J^1U; x^\lambda, \phi^A, \phi^A_\lambda)$ is any natural chart in $J^1\pi$ and ds is the local volume of $(U; x^\lambda)$.

The action of $\mathcal{L}_0$ is defined by

$$\mathcal{S} = \int_\Omega (j^1\sigma)^*\mathcal{L}_0 \tag{1.2}$$

where Ω is any compact domain in M with regular boundary $\partial\Omega$ and $\sigma \in \Gamma(\pi)$ is any local section (defined in an open subset containing Ω). One defines then the "first variation" $\delta\mathcal{S}$ of $\mathcal{S}$ by considering homotopic variations $\eta \equiv \delta\sigma \in \chi_V(\pi)$ with fixed values at the boundary $\partial\Omega$, $\chi_V(\pi)$ being the space of vertical vectorfields of π. The *critical sections* are those sections along which $\delta\mathcal{S}$ vanishes for any $\eta \in \chi_V(\pi)$ with fixed values at the boundary. They are characterised by the equation:

$$(j^2\sigma)^*[e(\mathcal{L}_0)] = 0 \tag{1.3}$$

which is called the *Euler–Lagrange equation.* Here $[e(\mathcal{L}_0)]$ is *Euler–Lagrange morphism*, a global bundle morphism $e(\mathcal{L}_0) : J^2\pi \to \Lambda^m(T^*M) \otimes V^*\pi$, where $V^*\pi$ is the dual bundle of the vertical vector bundle $V\pi$, locally defined by:

$$e(\mathcal{L}_0) = e_A(\mathcal{L}_0) ds \otimes d\phi^A \tag{1.4}$$

The first variation of $\mathcal{L}_0$ is in fact globally defined through a further global bundle morphism $f(\mathcal{L}_0) : J^1\pi \to \Lambda^{m-1}(T^*M) \otimes V^*\pi$, locally expressed by:

$$(j^1\sigma)^* f(\mathcal{L}_0) = (p^\mu_A(\mathcal{L}_0) \circ j^1\sigma) ds_\mu \otimes d\phi^A \tag{1.5}$$

where ds_μ is the surface element of $\partial\Omega$, defined so that $ds_\mu \wedge dx^\mu = (-1)^\mu ds$. The following holds for $T\mathcal{L}_0$:

$$(j^1\sigma)^*[T\mathcal{L}_0(\eta)] = (j^2\sigma)^* \langle e(\mathcal{L}_0)\,|\eta\rangle + (j^1\sigma)^* d\langle f(\mathcal{L}_0)\,|\,\eta\rangle \tag{1.6}$$

for any local section σ and any vertical vectorfield η which projects onto σ. Here and in the sequel $\langle\,|\,\rangle$ denotes standard duality between forms and vectorfields. Equation (1.6) is called the *global first variation formula* of $\mathcal{L}_0$. The integral counterpart of eqn. (1.6) is thence the following:

$$\delta\mathcal{S} = \int_\Omega \langle e(\mathcal{L}_0)|\eta\rangle + \int_{\partial\Omega} \langle f(\mathcal{L}_0)|\eta\rangle \tag{1.7}$$

from which the Euler–Lagrange equations (1.3) follow after imposing the appropriate boundary condition $\eta_{|\partial\Omega} = 0$.

In order to study the stability properties of critical sections, *i.e.* of the solutions of the Euler–Lagrange equation (1.3), one considers next the variation of the action under second–order deformations of σ. The second variation of $\mathcal{S}$ is then given by (see, *e.g.*, [5]):

$$\delta^2\mathcal{S} = \frac{1}{2}\left[\int_\Omega \langle e(\mathcal{L}_0)|\rho\rangle + \int_\Omega Hess_{\mathcal{L}_0}(j^1\eta) + \int_{\partial\Omega} \langle f(\mathcal{L}_0)|\rho\rangle\right] \tag{1.8}$$

where $\rho = \delta^2\sigma$ denotes the second variation of σ and the n-form $Hess_{\mathcal{L}_0}(j^1\eta)$ is the *Hessian* of $\mathcal{L}_0$. Locally eqn. (1.8) reads as follows:

$$\begin{aligned}\delta^2\mathcal{S} = \frac{1}{2}\Bigg[&\int_\Omega \left[\frac{\partial\mathcal{L}_0}{\partial\phi^A}\rho^A + \frac{\partial\mathcal{L}_0}{\partial{\phi^A}_\mu}{\rho^A}_\mu\right] ds \\ &+ \int_\Omega \left[\frac{\partial^2\mathcal{L}_0}{\partial\phi^A\partial\phi^B}\eta^A\eta^B + 2\frac{\partial^2\mathcal{L}_0}{\partial\phi^A\partial{\phi^B}_\mu}\eta^A{\eta^B}_\mu + \frac{\partial^2\mathcal{L}_0}{\partial{\phi^A}_\mu{\phi^B}_\nu}{\eta^A}_\mu{\eta^B}_\nu\right] ds\Bigg]\end{aligned} \tag{1.9}$$

A second order equation for η along critical sections, called the *(generalised) Jacobi equation*, can be obtained from (1.8) by suitable integrations by parts on $Hess$. In fact, as discussed *e.g.* in [5], eq. (1.8) can be conveniently rewritten as follows

$$\delta^2\mathcal{S} = \frac{1}{2}\left[\int_\Omega \langle e(\mathcal{L}_0)|\rho\rangle + \int_\Omega \langle Jac_{\mathcal{L}_0}(j^2\eta)|\eta\rangle + \int_{\partial\Omega} \hat{f}(\mathcal{L}_0)(\eta,\rho)\right] \tag{1.10}$$

where $\hat{f}(\mathcal{L}_0)(\eta,\rho)$ is a new boundary term depending both on η and ρ and the term $Jac_{\mathcal{L}_0}(j^2\eta)$ is locally given by

$$Jac_{\mathcal{L}_0}(j^2\eta) = J_A(j^2\eta)d\phi^A$$

with

$$\begin{aligned} J_A(j^2\eta) = &\frac{\partial^2\mathcal{L}_0}{\partial\phi^A\partial\phi^B}\,\eta^B + \frac{\partial^2\mathcal{L}_0}{\partial\phi^A\partial\phi^B{}_\mu}\,\eta^B{}_\mu \\ &- \frac{d}{dx^\mu}\left(\frac{\partial^2\mathcal{L}_0}{\partial\phi^A{}_\mu\partial\phi^B}\,\eta^B + \frac{\partial^2\mathcal{L}_0}{\partial\phi^A{}_\mu\partial\phi^B{}_\nu}\,\eta^B{}_\nu\right) \end{aligned} \tag{1.11}$$

Equating $J_A(j^2\eta)$ to zero, *i.e.* setting

$$Jac_{\mathcal{L}_0}(j^2\eta) = 0 \tag{1.12}$$

gives rise to the standard form of the *(generalized) Jacobi equation.*

Recalling that there is a natural bundle isomorphism $\mathcal{I} : V(J^1\pi) \to J^1(V\pi)$, locally defined by:

$$(x^\lambda, y^i, y^i_\mu; \eta^i, \eta^i_\mu) \mapsto (x^\lambda, y^i, \eta^i; y^i_\mu, \eta^i_\mu) \tag{1.13}$$

the first variation $\delta\mathcal{L}_0 : V(J^1\pi) \to \Lambda^n(T^*M)$ of $\mathcal{L}_0$, locally defined by:

$$\delta\mathcal{L}_0 \equiv \frac{\partial\mathcal{L}_0}{\partial y^i}\,\eta^i + \frac{\partial\mathcal{L}_0}{\partial y^i{}_\mu}\,\eta^i{}_\mu \tag{1.14}$$

defines in fact a new Lagrangian density in the bundle $V\pi$ by the prescription:

$$\mathcal{L}_1 = \delta\mathcal{L}_0 \circ (\mathcal{I})^{-1} \tag{1.15}$$

The new Lagrangian $\mathcal{L}_1$ is called the *first–order deformed Lagrangian.* It is not difficult to prove the following:

Theorem 1 *Let (B, M, π) be any fibered manifold and $\mathcal{L}_0$ be any first order Lagrangian density on π. Then the system formed by the Euler–Lagrange equation (1.3) and the Jacobi equation (1.12) of $\mathcal{L}_0$ is equivalent to the Euler–Lagrange equations of $\mathcal{L}_1$.*

This result, already used in [11] for the purpose of application to Riemannian Geometry, will not be deduced here. It will in fact follow from the analogous statement for second–order Lagrangians which will be discussed in the subsequent Sections of this paper.

2. Field Theory

We discuss now the generalisation of these results, including the Jacobi equation as developed in [5], for general second–order field theories. For the sake of immediate and easier understanding, the discussion will be given first in a coordinate language and using the analytical technique of truncated power expansions to express homotopic variations. Intrinsic geometric expressions will also be briefly discussed. The reader is referred to [5] for further details about notation.

2.1. Series Development of the Action and Euler–Lagrange Equations

Let us start by considering an action

$$\mathcal{S} = \int_\Omega \mathcal{L}_0(x,\, \phi,\, \partial\phi,\, \partial^2\phi)ds \tag{2.1}$$

and let us consider an infinitesimal expansion of the action $\mathcal{S}$ around a section $\phi^A = {\phi^A}_0$. The infinitesimal deformations of ϕ^A together with its derivatives ${\phi^A}_\mu$ and ${\phi^A}_{\mu\nu}$ are given by

$$\begin{aligned}
\phi^A &= {\phi^A}_0 + \epsilon\,\eta^A + \frac{1}{2}\,\epsilon^2\,\rho^A + o(\epsilon^3)\\
{\phi^A}_\mu &= ({\phi^A}_0)_\mu + \epsilon\,{\eta^A}_\mu + \frac{1}{2}\,\epsilon^2\,{\rho^A}_\mu + o(\epsilon^3)\\
{\phi^A}_{\mu\nu} &= ({\phi^A}_0)_{\mu\nu} + \epsilon\,{\eta^A}_{\mu\nu} + \frac{1}{2}\,\epsilon^2\,{\rho^A}_{\mu\nu} + o(\epsilon^3)
\end{aligned} \tag{2.2}$$

where ϵ is a smallness parameter, while $\eta = \eta^A\partial_A$ and $\rho = \rho^A\partial_A$ are vertical vectorfields which correspond to the "classical" first and second variation $\delta\phi$ and $\delta^2\phi$ respectively. The variation of the action, at the second–order in ϵ, is given by

$$\mathcal{S}(\epsilon) = \mathcal{S}_0 + \epsilon\,\delta\mathcal{S} + \frac{1}{2}\,\epsilon^2\,(\delta^2\mathcal{S}) + \cdots \tag{2.3}$$

where

$$\delta\mathcal{S} = \int_\Omega \left[\frac{\partial\mathcal{L}_0}{\partial\phi^A}\,\eta^A + \frac{\partial\mathcal{L}_0}{\partial{\phi^A}_\mu}\,{\eta^A}_\mu + \frac{\partial\mathcal{L}_0}{\partial{\phi^A}_{\mu\nu}}\,{\eta^A}_{\mu\nu}\right] ds \tag{2.4a}$$

$$\delta^2\mathcal{S} = \int_\Omega \left[\frac{\delta\mathcal{L}_0}{\delta\phi^A}\,\rho^A + \frac{\delta\mathcal{L}_0}{\delta{\phi^A}_\mu}\,{\rho^A}_\mu + \frac{\delta\mathcal{L}_0}{\delta{\phi^A}_{\mu\nu}}\,{\rho^A}_{\mu\nu}\right] ds + \int_\Omega \mathcal{L}_2\,ds \tag{2.4b}$$

and

$$
\begin{aligned}
&\mathcal{L}_2 = \\
&\frac{\partial^2 \mathcal{L}_0}{\partial \phi^A \partial \phi^B}\, \eta^A\, \eta^B + 2\, \frac{\partial^2 \mathcal{L}_0}{\partial \phi^A \partial {\phi^B}_{\mu}}\, \eta^A\, {\eta^B}_{\mu} + 2\, \frac{\partial^2 \mathcal{L}_0}{\partial \phi^A \partial {\phi^B}_{\mu\nu}}\, \eta^A\, {\eta^B}_{\mu\nu} \\
&+ \frac{\partial^2 \mathcal{L}_0}{\partial {\phi^A}_{\mu} \partial {\phi^B}_{\nu}}\, {\eta^A}_{\mu}\, {\eta^B}_{\nu} + 2\, \frac{\partial^2 \mathcal{L}_0}{\partial {\phi^A}_{\mu} \partial {\phi^B}_{\nu\lambda}}\, {\eta^A}_{\mu}\, {\eta^B}_{\nu\lambda} + \frac{\partial^2 \mathcal{L}_0}{\partial {\phi^A}_{\mu\nu} \partial {\phi^B}_{\lambda\rho}}\, {\eta^A}_{\mu\nu}\, {\eta^B}_{\lambda\rho}
\end{aligned}
\tag{2.5}
$$

is the Hessian of $\mathcal{L}_0$, which by later convenience we have denoted by $\mathcal{L}_2$.

With the expansion above, we can now deal with the problem of extremality and stability of the action. Hereafter, the original Lagrangian $\mathcal{L}_0$ will be simply denoted by $\mathcal{L}$ to simplify notation.

Let us then assume that $\phi^A = {\phi^A}_0(x)$ is a section along which $\mathcal{S}$ has an extremum, *i.e.* the first variation of the action vanishes for all first–order deformations η^A. To this purpose it is convenient to perform an integration by parts and rewrite $\delta\mathcal{S}$ as follows:

$$
\delta\mathcal{S} = \int_{\Omega} \frac{\delta \mathcal{L}}{\delta \phi^A}\, \eta^A\, ds + \oint_{\partial\Omega} \left[\frac{\delta \mathcal{L}}{\delta {\phi^A}_{\mu}}\, \eta^A + \frac{\partial \mathcal{L}}{\partial {\phi^A}_{\mu\nu}}\, {\eta^A}_{\nu} \right] ds_{\mu} \tag{2.6}
$$

where

$$
\begin{aligned}
\frac{\delta \mathcal{L}}{\delta \phi^A} &= \frac{\partial \mathcal{L}}{\partial \phi^A} - \frac{d}{dx^{\mu}} \left(\frac{\partial \mathcal{L}}{\partial {\phi^A}_{\mu}} \right) + \frac{d^2}{dx^{\mu} dx^{\nu}} \left(\frac{\partial \mathcal{L}}{\partial {\phi^A}_{\mu\nu}} \right) = \frac{\partial \mathcal{L}}{\partial \phi^A} - \frac{d}{dx^{\mu}} \left(\frac{\delta \mathcal{L}}{\delta {\phi^A}_{\mu}} \right) \\
\frac{\delta \mathcal{L}}{\delta {\phi^A}_{\mu}} &= \frac{\partial \mathcal{L}}{\partial {\phi^A}_{\mu}} - \frac{d}{dx^{\nu}} \left(\frac{\partial \mathcal{L}}{\partial {\phi^A}_{\mu\nu}} \right)
\end{aligned}
\tag{2.7}
$$

are the Euler–Lagrange derivatives.

As in the first–order case (see [11]), the local coordinate expression (2.6) corresponds to the intrinsic decomposition

$$
\delta\mathcal{S} = \int_{\Omega} \langle e(\mathcal{L}) \,|\, \eta \rangle + \int_{\partial\Omega} \langle f(\mathcal{L}) \,|\, j^1 \eta \rangle \tag{2.8}
$$

where $e(\mathcal{L}) : J^4\pi \to \Lambda^m(T^*M) \otimes V^*\pi$ is the Euler–Lagrange morphism and $f(\mathcal{L}) : J^3\pi \to \Lambda^{m-1}(T^*M) \otimes V^*\pi^1$ is the canonical Poincarè–Cartan morphism (uniquely existing for second–order theories, as is well known from the general theory; see [12], [13] and ref.s quoted therein).

The extremality of the action is classically described by the Euler–Lagrange equations

$$
\frac{\delta \mathcal{L}}{\delta \phi^A} = 0 \tag{2.9}
$$

under the boundary conditions

$$\eta^A\Big|_{\partial\Omega} = 0 \tag{2.10a}$$

$$\eta^A{}_\mu\Big|_{\partial\Omega} = 0 \tag{2.10b}$$

which just mean that the deformation of the local section ϕ_0 is fixed at the boundary, together with its first derivatives. The above conditions just concern the extremality of the action. They do not say anything about the stability of the solutions and to this purpose we must go to the next order of deformation.

2.2. The Jacobi Equation

A criterion for the stability of the Euler–Lagrange equations is obtained by looking at the sign of the second variation $\delta^2\mathcal{S}$. If $\delta^2\mathcal{S} > 0$ we have a minimum; if $\delta^2\mathcal{S} < 0$ we have a maximum; if $\delta^2\mathcal{S} = 0$ we might have a degenerate critical point. Assuming that we are "on shell", *i.e.* that the Euler–Lagrange equations hold along the section ϕ_0, then $\delta^2\mathcal{S}$ reduces to

$$\delta^2\mathcal{S} = \oint_{\partial\Omega}\left(\frac{\delta\mathcal{L}}{\delta\phi^A{}_\mu}\,\rho^A + \frac{\partial\mathcal{L}}{\partial\phi^A{}_{\mu\nu}}\,\rho^A{}_\nu\right) ds_\mu + \int_\Omega \mathcal{L}_2\, ds \tag{2.11}$$

In order for $\delta^2\mathcal{S}$ to have a definite sign we impose the further boundary conditions

$$\begin{aligned} \rho^A\Big|_{\partial\Omega} &= 0\,, \\ \rho^A{}_\mu\Big|_{\partial\Omega} &= 0 \end{aligned} \tag{2.12}$$

i.e., we require that ϕ_0 is stronly fixed at the boundary up to order two. Under these stronger conditions $\delta^2\mathcal{S}$ reduces to

$$\delta^2\mathcal{S} = \int_\Omega \mathcal{L}_2\, ds \tag{2.13}$$

As already discussed in [5] an equation called the Jacobi equation can now be obtained by a suitable integration by parts. Let us however remark that the quadratic form appearing in (2.13) can be changed by adding surface terms which, due to conditions (2.10) vanish at the integration boundary. Therefore, the form of the Jacobi equation to which one arrives strongly depends on the terms added at the boundary. One criteria would be to add surface terms which do not change the dependence on $(\eta, \partial\eta, \partial^2\eta)$. For this let us rewrite $\mathcal{L}_2$ in the form

$$
\begin{aligned}
&\mathcal{L}_2 = \\
&\frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial \phi^B}\, \eta^A\, \eta^B + \frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial {\phi^B}_\mu}\, \eta^A\, {\eta^B}_\mu + \frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial {\phi^B}_{\mu\nu}}\, \eta^A\, {\eta^B}_{\mu\nu} \\
&+ \frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_\mu \partial \phi^B}\, {\eta^A}_\mu\, \eta^B + \frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_\mu \partial {\phi^B}_\nu}\, {\eta^A}_\mu\, {\eta^B}_\nu + \frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_\mu \partial {\phi^B}_{\nu\lambda}}\, {\eta^A}_\mu\, {\eta^B}_{\nu\lambda} \\
&+ \frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_{\mu\nu} \partial \phi^B}\, {\eta^A}_{\mu\nu}\, \eta^B + \frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_{\mu\nu} \partial {\phi^B}_\lambda}\, {\eta^A}_{\mu\nu}\, {\eta^B}_\lambda + \frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_{\mu\nu} \partial {\phi^B}_{\lambda\rho}}\, {\eta^A}_{\mu\nu}\, {\eta^B}_{\lambda\rho}
\end{aligned}
\tag{2.14}
$$

The only terms which can be integrated by parts are the fourth, fifth, seventh and eighth terms. The four possible outcomes are summarised hereafter:

$$
\begin{aligned}
\mathcal{L}_2 \cong{}& \frac{\partial}{\partial \phi^A}\left(\frac{\partial \mathcal{L}}{\partial \phi^B} - \frac{d}{dx^\mu}\left(\frac{\partial \mathcal{L}}{\partial {\phi^B}_\mu}\right)\right) \eta^A\, \eta^B + \left(\frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial {\phi^B}_\mu} - \frac{\partial^2 \mathcal{L}}{\partial \phi^B \partial {\phi^A}_\mu}\right) \eta^A\, {\eta^B}_\mu \\
&+ 2\, \frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial {\phi^B}_{\mu\nu}}\, \eta^A\, {\eta^B}_{\mu\nu} + \frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_\mu \partial {\phi^B}_\nu}\, {\eta^A}_\mu\, {\eta^B}_\nu + 2\, \frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_\mu \partial {\phi^B}_{\nu\lambda}}\, {\eta^A}_\mu\, {\eta^B}_{\nu\lambda} \\
&+ \frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_{\mu\nu} \partial {\phi^B}_{\lambda\rho}}\, {\eta^A}_{\mu\nu}\, {\eta^B}_{\lambda\rho} + \frac{d}{dx^\mu}\left(\frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial {\phi^B}_\mu}\, \eta^A\, \eta^B\right)
\end{aligned}
\tag{2.15a}
$$

$$
\begin{aligned}
\mathcal{L}_2 \cong{}& \frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial \phi^B}\, \eta^A\, \eta^B + \left(2\, \frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial {\phi^B}_\mu} - \frac{d}{dx^\nu}\left(\frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_\mu \partial {\phi^B}_\nu}\right)\right) \eta^A\, {\eta^B}_\mu \\
&+ \left(2\, \frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial {\phi^B}_{\mu\nu}} - \frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_\mu \partial {\phi^B}_\nu}\right) \eta^A\, {\eta^B}_{\mu\nu} + 2\, \frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_\mu \partial {\phi^B}_{\nu\lambda}}\, {\eta^A}_\mu\, {\eta^B}_{\nu\lambda} \\
&+ \frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_{\mu\nu} \partial {\phi^B}_{\lambda\rho}}\, {\eta^A}_{\mu\nu}\, {\eta^B}_{\lambda\rho} + \frac{d}{dx^\mu}\left(\frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_\mu \partial {\phi^B}_\nu}\, \eta^A\, {\eta^B}_\nu\right)
\end{aligned}
\tag{2.15b}
$$

$$
\begin{aligned}
\mathcal{L}_2 \cong{}& \frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial \phi^B}\, \eta^A\, \eta^B + \left(2\, \frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial {\phi^B}_\mu} - \frac{\partial}{\partial \phi^A} \frac{d}{dx^\nu}\left(\frac{\partial \mathcal{L}}{\partial {\phi^B}_{\mu\nu}}\right)\right) \eta^A\, {\eta^B}_\mu \\
&+ \frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial {\phi^B}_{\mu\nu}}\, \eta^A\, {\eta^B}_{\mu\nu} + \left(\frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_\mu \partial {\phi^B}_\nu} - \frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial {\phi^B}_{\mu\nu}}\right) {\eta^A}_\mu\, {\eta^B}_\nu \\
&+ 2\, \frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_\mu \partial {\phi^B}_{\nu\lambda}}\, {\eta^A}_\mu\, {\eta^B}_{\nu\lambda} + \frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_{\mu\nu} \partial {\phi^B}_{\lambda\rho}}\, {\eta^A}_{\mu\nu}\, {\eta^B}_{\lambda\rho} \\
&+ \frac{d}{dx^\mu}\left(\frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial {\phi^B}_{\mu\nu}}\, \eta^A\, {\eta^B}_\nu\right)
\end{aligned}
\tag{2.15c}
$$

$$
\begin{aligned}
\mathcal{L}_2 \cong\ & \frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial \phi^B} \eta^A \eta^B + 2 \frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial {\phi^B}_\mu} \eta^A {\eta^B}_\mu + 2 \frac{\partial^2 L}{\partial \phi^A \partial {\phi^B}_{\mu\nu}} \eta^A {\eta^B}_{\mu\nu} \\
& + \left(\frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_\mu \partial {\phi^B}_\nu} - \frac{d}{dx^\lambda} \left(\frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_\mu \partial {\phi^B}_{\nu\lambda}} \right) \right) {\eta^A}_\mu {\eta^B}_\nu \\
& + \left(\frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_\mu \partial {\phi^B}_{\nu\lambda}} - \frac{\partial^2 \mathcal{L}}{\partial {\phi^B}_\mu \partial {\phi^A}_{\nu\lambda}} \right) {\eta^A}_\mu {\eta^B}_{\nu\lambda} + \frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_{\mu\nu} \partial {\phi^B}_{\lambda\rho}} {\eta^A}_{\mu\nu} {\eta^B}_{\lambda\rho} \\
& + \frac{d}{dx^\mu} \left(\frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_{\mu\nu} \partial {\phi^B}_\lambda} {\eta^A}_\nu {\eta^B}_\lambda \right)
\end{aligned}
\tag{2.15d}
$$

where $\cong$ means equal modulo a total divergence.

Apart from first–order divergencies, we can also add second–order divergencies to the Hessian. The only two possibilities which do not introduce derivatives higher than fourth–order ones are the following:

$$
\begin{aligned}
& \frac{d^2}{dx^\mu dx^\nu} \left(\frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_\mu \partial {\phi^B}_\nu} \eta^A \eta^B \right) = \\
& \frac{d^2}{dx^\mu dx^\nu} \left(\frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_\mu \partial {\phi^B}_\nu} \right) \eta^A \eta^B + 4 \frac{d}{dx^\mu} \left(\frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_\mu \partial {\phi^B}_\nu} \right) \eta^A {\eta^B}_\nu \\
& + 2 \frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_\mu \partial {\phi^B}_\nu} \eta^A {\eta^B}_{\mu\nu} + 2 \frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_\mu \partial {\phi^B}_\nu} {\eta^A}_\mu {\eta^B}_\nu
\end{aligned}
$$

$$
\begin{aligned}
& \frac{d^2}{dx^\mu dx^\nu} \left(\frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial {\phi^B}_{\mu\nu}} \eta^A \eta^B \right) = \\
& \frac{d^2}{dx^\mu dx^\nu} \left(\frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial {\phi^B}_{\mu\nu}} \right) \eta^A \eta^B + 2 \frac{d}{dx^\mu} \left(\frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial {\phi^B}_{\mu\nu}} + \frac{\partial^2 \mathcal{L}}{\partial \phi^B \partial {\phi^A}_{\mu\nu}} \right) \eta^A {\eta^B}_\nu \\
& + \left(\frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial {\phi^B}_{\mu\nu}} + \frac{\partial^2 \mathcal{L}}{\partial \phi^B \partial {\phi^A}_{\mu\nu}} \right) \eta^A {\eta^B}_{\mu\nu} + 2 \frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial {\phi^B}_{\mu\nu}} {\eta^A}_\mu {\eta^B}_\nu
\end{aligned}
\tag{2.16}
$$

A simpler possibility is to combine all the above integration by parts in a single decomposition displaying better properties. The result similar to the analogous one for first–order theories (see [5]) is given by

$$
\begin{aligned}
\mathcal{L}_2 \cong & \frac{\partial}{\partial \phi^A}\left(\frac{\delta \mathcal{L}}{\delta \phi^B}\right) \eta^A \eta^B + \left(\frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial \phi^B{}_\mu} - \frac{\partial^2 \mathcal{L}}{\partial \phi^B \partial \phi^A{}_\mu}\right) \eta^A \eta^B{}_\mu \\
& + \left(\frac{\partial^2 \mathcal{L}}{\partial \phi^A{}_\mu \partial \phi^B{}_\nu} - \frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial \phi^B{}_{\mu\nu}}\right) \eta^A{}_\mu \eta^B{}_\nu \\
& + \left(\frac{\partial^2 \mathcal{L}}{\partial \phi^A{}_\mu \partial \phi^B{}_{\nu\lambda}} - \frac{\partial^2 \mathcal{L}}{\partial \phi^B{}_\mu \partial \phi^A{}_{\nu\lambda}}\right) \eta^A{}_\mu \eta^B{}_{\nu\lambda} + \frac{\partial^2 \mathcal{L}}{\partial \phi^A{}_{\mu\nu} \partial \phi^B{}_{\lambda\rho}} \eta^A{}_{\mu\nu} \eta^B{}_{\lambda\rho} \\
& + \frac{d}{dx^\mu}\left(\frac{\partial}{\partial \phi^A} \frac{\delta \mathcal{L}}{\delta \phi^B{}_\mu} \eta^A \eta^B + \frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial \phi^B{}_{\mu\nu}} \eta^A \eta^B{}_\nu + \frac{\partial^2 \mathcal{L}}{\partial \phi^A{}_\mu \partial \phi^B{}_{\nu\lambda}} \eta^A{}_\nu \eta^B{}_\lambda\right)
\end{aligned} \tag{2.17}
$$

However, other terms can be added to the Hessian changing the dependence on only $(\eta, \partial\eta, \partial^2\eta)$. An interesting one is the following further expression:

$$
\begin{aligned}
\mathcal{L}_2 \cong & \Bigg[\frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial \phi^B} \eta^B + \frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial \phi^B{}_\lambda} \eta^B{}_\lambda + \frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial \phi^B{}_{\lambda\rho}} \eta^B{}_{\lambda\rho} \\
& - \frac{d}{dx^\mu}\left(\frac{\partial^2 \mathcal{L}}{\partial \phi^A{}_\mu \partial \phi^B} \eta^B + \frac{\partial^2 \mathcal{L}}{\partial \phi^A{}_\mu \partial \phi^B{}_\lambda} \eta^B{}_\lambda + \frac{\partial^2 \mathcal{L}}{\partial \phi^A{}_\mu \partial \phi^B{}_{\lambda\rho}} \eta^B{}_{\lambda\rho}\right) \\
& + \frac{d^2}{dx^\mu dx^\nu}\left(\frac{\partial^2 \mathcal{L}}{\partial \phi^A{}_{\mu\nu} \partial \phi^B} \eta^B + \frac{\partial^2 \mathcal{L}}{\partial \phi^A{}_{\mu\nu} \partial \phi^B{}_\lambda} \eta^B{}_\lambda + \frac{\partial^2 \mathcal{L}}{\partial \phi^A{}_{\mu\nu} \partial \phi^B{}_{\lambda\rho}} \eta^B{}_{\lambda\rho}\right) \Bigg] \eta^A \\
& - \frac{d}{dx^\mu}\Bigg[\Bigg(\frac{\partial^2 \mathcal{L}}{\partial \phi^A{}_\mu \partial \phi^B} \eta^B + \frac{\partial^2 \mathcal{L}}{\partial \phi^A{}_\mu \partial \phi^B{}_\lambda} \eta^B{}_\lambda + \frac{\partial^2 \mathcal{L}}{\partial \phi^A{}_\mu \partial \phi^B{}_{\lambda\rho}} \eta^B{}_{\lambda\rho} \\
& - \frac{d}{dx^\nu}\left(\frac{\partial^2 \mathcal{L}}{\partial \phi^A{}_{\mu\nu} \partial \phi^B} \eta^B + \frac{\partial^2 \mathcal{L}}{\partial \phi^A{}_{\mu\nu} \partial \phi^B{}_\lambda} \eta^B{}_\lambda + \frac{\partial^2 \mathcal{L}}{\partial \phi^A{}_{\mu\nu} \partial \phi^B{}_{\lambda\rho}} \eta^B{}_{\lambda\rho}\right) \Bigg) \eta^A \\
& + \left(\frac{\partial^2 \mathcal{L}}{\partial \phi^A{}_{\mu\nu} \partial \phi^B} \eta^B + \frac{\partial^2 \mathcal{L}}{\partial \phi^A{}_{\mu\nu} \partial \phi^B{}_\lambda} \eta^B{}_\lambda + \frac{\partial^2 \mathcal{L}}{\partial \phi^A{}_{\mu\nu} \partial \phi^B{}_{\lambda\rho}} \eta^B{}_{\lambda\rho}\right) \eta^A{}_\nu \Bigg]
\end{aligned} \tag{2.18}
$$

In all the above cases the surface terms cancel at the integration boundary. Therefore, different Jacobi equations would be obtained. The equation ensuing from (2.18), namely

$$
\begin{aligned}
&\frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial \phi^B}\,\eta^B + \frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial {\phi^B}_\lambda}\,{\eta^B}_\lambda + \frac{\partial^2 \mathcal{L}}{\partial \phi^A \partial {\phi^B}_{\lambda\rho}}\,{\eta^B}_{\lambda\rho} \\
&-\frac{d}{dx^\mu}\left(\frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_\mu \partial \phi^B}\,\eta^B + \frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_\mu \partial {\phi^B}_\lambda}\,{\eta^B}_\lambda + \frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_\mu \partial {\phi^B}_{\lambda\rho}}\,{\eta^B}_{\lambda\rho}\right) \\
&+\frac{d^2}{dx^\mu dx^\nu}\left(\frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_{\mu\nu} \partial \phi^B}\,\eta^B + \frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_{\mu\nu} \partial {\phi^B}_\lambda}\,{\eta^B}_\lambda + \frac{\partial^2 \mathcal{L}}{\partial {\phi^A}_{\mu\nu} \partial {\phi^B}_{\lambda\rho}}\,{\eta^B}_{\lambda\rho}\right) = 0
\end{aligned}
\tag{2.19}
$$

will be called the "standard Jacobi equation" for $\mathcal{L}$. It is the equation (0.2) we already mentioned in the Introduction.

3. Hierarchical Structure of the Deformed Lagrangian

Now we introduce a hierarchical structure associated to the deformations of the Lagrangian.

3.1. The Deformed Lagrangian

Let us again consider an infinitesimal deformation of the field ϕ^A and its derivatives ${\phi^A}_\mu$ and ${\phi^A}_{\mu\nu}$, up to second–order in ϵ. Then, the variation of the Lagrangian, at second–order in ϵ, is given by

$$\mathcal{L}(\eta) = \mathcal{L}_0 + \epsilon\, \mathcal{L}_1(\eta) + \frac{1}{2}\,\epsilon^2\,\left[\mathcal{L}_2(\eta) + \mathcal{L}_1(\rho)\right] \tag{3.1}$$

where $\mathcal{L}_1$ is given by

$$\mathcal{L}_1(\eta) = \frac{\partial \mathcal{L}_0}{\partial \phi^A}\,\eta^A + \frac{\partial \mathcal{L}_0}{\partial {\phi^A}_\mu}\,{\eta^A}_\mu + \frac{\partial \mathcal{L}_0}{\partial {\phi^A}_{\mu\nu}}\,{\eta^A}_{\mu\nu} \tag{3.2}$$

while $\mathcal{L}_2$ is given by eq.n (2.5).

The introduction of the above definitions is not only a matter of notational convenience since $\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{L}_2$ satisfy a series of remarkable identities. In fact, it can be easily checked that the Euler–Lagrange derivatives of $\mathcal{L}_1$ and $\mathcal{L}_2$ are given by

$$
\begin{aligned}
\frac{\delta \mathcal{L}_1}{\delta \eta^A} &= \frac{\delta \mathcal{L}_0}{\delta \phi^A} \\
\frac{\delta \mathcal{L}_2}{\delta \eta^A} &= \frac{\delta \mathcal{L}_1}{\delta \phi^A}
\end{aligned}
\tag{3.3}
$$

Moreover, it can be easily checked (see also [4]) that the equation

$$\frac{\delta \mathcal{L}_2}{\delta \eta^A} = 0 \tag{3.4}$$

corresponds to the Jacobi equation (2.19) in its standard form.

3.2. Unified description of the Euler-Lagrange and Jacobi equations

Let us now remark that in virtue of (3.3) the Euler–Lagrange equation and the Jacobi equation can be rewritten together as

$$\begin{aligned} \frac{\delta \mathcal{L}_0}{\delta \phi^A} &= \frac{\delta \mathcal{L}_1}{\delta \eta^A} = 0 \\ \frac{\delta \mathcal{L}_2}{\delta \eta^A} &= \frac{\delta \mathcal{L}_1}{\delta \phi^A} = 0 \end{aligned} \tag{3.5}$$

Therefore, we can consider the above system of equations as coming from a single variational principle based on the deformed Lagrangian $\mathcal{L}_1$ where, however, the dynamical variables have been "doubled" to ϕ's and η's.

We can thence introduce momenta canonically conjugated to ϕ's and η's by means of the standard prescriptions

$$\begin{aligned} \pi_A{}^\mu &= \frac{\partial \mathcal{L}_1}{\partial \phi^A{}_\mu} - \frac{d}{dx^\nu}\left(\frac{\partial \mathcal{L}_1}{\partial \phi^A{}_{\mu\nu}}\right) \\ N_A{}^{\mu\nu} &= \frac{\partial \mathcal{L}_1}{\partial \phi^A{}_{\mu\nu}} \\ p_A{}^\mu &= \frac{\partial \mathcal{L}_1}{\partial \eta^A{}_\mu} - \frac{d}{dx^\nu}\left(\frac{\partial \mathcal{L}_1}{\partial \eta^A{}_{\mu\nu}}\right) \\ n_A{}^{\mu\nu} &= \frac{\partial \mathcal{L}_1}{\partial \eta^A{}_{\mu\nu}} \end{aligned} \tag{3.6}$$

and a canonical energy–momentum tensor by setting:

$$\mathcal{H}^\mu{}_\nu = \pi_A{}^\mu\, \phi^A{}_\nu + N_A{}^{\mu\lambda}\, \phi^A{}_{\lambda\nu} + p_A{}^\mu\, \eta^A{}_\nu + n_A{}^{\mu\lambda}\, \eta^A{}_{\lambda\nu} - \delta^\mu_\nu\, \mathcal{L}_1 \tag{3.7}$$

The explicit expression of $\mathcal{H}^\mu{}_\nu$ for our case is not relevant here. What is really important is the fact that, if $\mathcal{L}_0$ does not depend on "space–time variables" x^λ, then $\mathcal{H}^\mu{}_\nu$ is a conserved quantity. Therefore, the allowed deformations are selected by the first–order condition

$$\frac{d\mathcal{H}^\mu{}_\nu}{dx^\mu} = 0 \tag{3.8}$$

which can thence replace the second–order condition given by the Jacobi equation (2.19).

Let us also remark that the Hessian matrix associated to $\mathcal{L}_1$ is given by

$$W(\mathcal{L}_1) = \begin{pmatrix} \frac{\partial^2 \mathcal{L}_1}{\partial \phi^A{}_\mu \partial \phi^B{}_\nu} & \frac{\partial^2 \mathcal{L}_1}{\partial \phi^A{}_\mu \partial \eta^B{}_\nu} \\ \frac{\partial^2 \mathcal{L}_1}{\partial \eta^A{}_\mu \partial \phi^B{}_\nu} & \frac{\partial^2 \mathcal{L}_1}{\partial \eta^A{}_\mu \partial \eta^B{}_\nu} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 \mathcal{L}_1}{\partial \phi^A{}_\mu \partial \phi^B{}_\nu} & W^{\mu\nu}_{AB}(\mathcal{L}_0) \\ W^{\mu\nu}_{AB}(\mathcal{L}_0) & 0 \end{pmatrix} \tag{3.9}$$

so that the following holds true

$$\Big| \det[W(\mathcal{L}_1)] \Big| = [\det[W(\mathcal{L}_0)]]^2 \tag{3.10}$$

which ensures that the regularity of $\mathcal{L}_1$ depends only on the regularity of $\mathcal{L}_0$.

Suitable globalisations of this concept can be achieved by introducing a connection (see, *e.g.*, [14],[15]). See also [16] and [17] for a discussion of so-called "d–invariance" of these energy–momentum tensors.

4. Examples and Applications

Here we consider some simple Lagrangians which involve up to second–order derivatives in order to illustrate our general results.

4.1. The Riewe Lagrangian

The Riewe Lagrangian (see [18]) was introduced to describe a classically spinning particle. It is given by

$$L = \frac{1}{2}\, m\, \dot{\mathbf{q}}^2 - \frac{1}{2}\, \frac{m}{\omega^2}\, \ddot{\mathbf{q}}^2 \tag{4.1}$$

where $\mathbf{q}$ is a three–dimensional vector. The equations of motion are given by

$$\frac{\delta L}{\delta q^i} = -m\, \delta_{ij} \left(\ddot{q}^j + \frac{1}{\omega^2}\, \ddot{\ddot{q}}^j \right) = 0 \tag{4.2}$$

The solutions of the above equation of motion are given by

$$q^i = q^i_0 + v^i_0\, t + a^i\, \cos(\omega\, t) + b^i\, \sin(\omega\, t) \tag{4.3}$$

These solutions describe a helical motion along a ellipsoid of semiaxis a^i and b^i oriented in the direction of v^i.

The corresponding Jacobi equation is given by

$$\ddot{\eta}^j + \frac{1}{\omega^2}\, \ddot{\ddot{\eta}}^j = 0 \tag{4.4}$$

4.2. The Shadwick Lagrangian

The Shadwick Lagrangian is given by:

$$\mathcal{L} = \phi \left(\phi_{00}\,\phi_{11} - \phi_{01}^2\right) \tag{4.5}$$

where ϕ is a scalar field (see [19]). The field equations are given by

$$\frac{\delta \mathcal{L}}{\delta \phi} = 3 \left(\phi_{00}\,\phi_{11} - {\phi_{01}}^2\right) = 0 \tag{4.6}$$

The corresponding Jacobi equation is given by

$$\phi_{11}\,\eta_{00} + \phi_{00}\,\eta_{11} - 2\,\phi_{01}\,\eta_{01} = 0 \tag{4.7}$$

4.3. The Klein-Gordon equation

The Klein-Gordon equation is obtained from the Lagrangian density

$$\mathcal{L}_0 = \frac{1}{2} \left(g^{\mu\nu}\,\phi_\mu\,\phi_\nu - m^2\,\phi^2\right) \tag{4.8}$$

whose first-order and second-order deformed Lagrangians are given by

$$\begin{aligned} \mathcal{L}_1 &= g^{\mu\nu}\,\phi_\mu\,\eta_\nu - m^2\,\phi\,\eta\,, \\ \mathcal{L}_2 &= \frac{1}{2} \left(g^{\mu\nu}\,\eta_\mu\,\eta_\nu - m^2\,\eta^2\right) \end{aligned} \tag{4.9}$$

The Euler-Lagrange and Jacobi equations are given by

$$\begin{aligned} \frac{\delta \mathcal{L}_1}{\delta \eta} &= -\left(g^{\mu\nu}\,\phi_{\mu\nu} + m^2\,\phi\right) = 0 \\ \frac{\delta \mathcal{L}_1}{\delta \phi} &= -\left(g^{\mu\nu}\,\eta_{\mu\nu} + m^2\,\eta\right) = 0 \end{aligned} \tag{4.10}$$

respectively. Solutions to these equations are easily expressed by

$$\begin{aligned} \phi(x,\,t) &= \int \left(f_+(k)\,e^{i(kx+\sqrt{k^2+m^2}t)} + f_-(k)\,e^{i(kx-\sqrt{k^2+m^2}t)}\right)\,dk \\ \eta(x,\,t) &= \int \left(h_+(k)\,e^{i(kx+\sqrt{k^2+m^2}t)} + h_-(k)\,e^{i(kx-\sqrt{k^2+m^2}t)}\right)\,dk \end{aligned} \tag{4.11}$$

such that

$$\phi_0(x,t) = i\int \sqrt{k^2+m^2}\,\left(f_+(k)\,e^{i(kx+\sqrt{k^2+m^2}t)} - f_-(k)\,e^{i(kx-\sqrt{k^2+m^2}t)}\right)\,dk$$
$$\phi_i(x,t) = i\int k_i\,\left(f_+(k)\,e^{i(kx+\sqrt{k^2+m^2}t)} + f_-(k)\,e^{i(kx-\sqrt{k^2+m^2}t)}\right)\,dk$$
$$\eta_0(x,t) = i\int \sqrt{k^2+m^2}\,\left(h_+(k)\,e^{i(kx+\sqrt{k^2+m^2}t)} - h_-(k)\,e^{i(kx-\sqrt{k^2+m^2}t)}\right)\,dk$$
$$\eta_i(x,t) = i\int k_i\,\left(h_+(k)\,e^{i(kx+\sqrt{k^2+m^2}t)} + h_-(k)\,e^{i(kx-\sqrt{k^2+m^2}t)}\right)\,dk \qquad (4.12)$$

The appropriate canonical energy-momentum tensor is thence given by

$$\mathcal{H}^{\mu}{}_{\nu}(\mathcal{L}_1) = \phi_\nu\, g^{\mu\lambda}\,\eta_\lambda + \eta_\nu\, g^{\mu\lambda}\,\phi_\lambda - \delta^\mu_\nu\,\mathcal{L}_1 \qquad (4.13)$$

and its time-time component $\mathcal{H}^0{}_0(\mathcal{L}_1)$ is given by

$$\mathcal{H}^0{}_0(\mathcal{L}_1) = \phi_0\,\eta_0 + \phi_i\,\eta_i + m^2\,\phi\,\eta \qquad (4.14)$$

The associated energy (see also [6],[7]) is therefore given by

$$E(\mathcal{L}_1) = \int \mathcal{H}^0{}_0(\mathcal{L}_1)\,dx \qquad (4.15)$$

and an evaluation on–shell gives immediately

$$E(\mathcal{L}_1) = 2\,(2\,\pi)^3\,m^2 \int (f_+(k)\,h_-(k) + f_-(k)\,h_+(k))\,dk = \text{constant} \qquad (4.16)$$

This condition is equivalent to the Jacobi equation.

4.4. Geodesics in a Riemannian manifold

A further example of application to standard Jacobi fields along geodesics of a Riemannian manifold (M,g) has been discussed in [11]. Here we can shortly summarise the results.

Let Q be a n–dimensional manifold and (TQ, Q, τ_Q) its tangent bundle. Let g be a Riemannian metric on Q. The *geodesics* of (Q,g) are those curves $\gamma : \mathbf{R} \to Q$ whose tangent vector $\dot\gamma$ is parallel along γ, *i.e.*, it satisfies $\nabla_{\dot\gamma}\dot\gamma = 0$; in local components:

$$\ddot q^i + \Gamma^i_{jk}\,\dot q^j\,\dot q^k = 0 \qquad (4.17)$$

The *Jacobi fields* of (M,g) are those vectorfields $\eta = \eta^i\,\partial_i$ defined along geodesics γ by the differential equation:

$$\nabla^2_{\dot\gamma}\eta + Riem(\eta, \dot\gamma, \dot\gamma) = 0 \tag{4.18}$$

where $\nabla^2_{\dot\gamma}$ denotes the second–order covariant derivative along the curve γ and $Riem$ is the tri–linear mapping defining the Riemannian curvature of g. Jacobi fields generically define infinitesimal deformations of geodesics into families of nearby geodesics. According to [20], the metric g can be lifted to a metric g^C on the manifold TQ, called the "complete lift", as follows. Let $g = g_{ij}dq^i dq^j$ in a local chart $(U; q^i)$; then the corresponding local expression of g^C in $(TU; q^i, u^i)$ is

$$g^C = 2g_{ij}\delta u^i dq^j \tag{4.19}$$

where δu^i denotes the following:

$$\delta u^i = du^i + \Gamma^i_{mk} u^m dq^k \tag{4.20}$$

For any function $f : Q \to \mathbf{R}$ a new function $\partial f : TQ \to \mathbf{R}$ is defined by setting locally:

$$(\partial f)(q^i, u^i) \equiv (\partial_j f) u^j \tag{4.21}$$

With this notation g^C can be locally expressed by:

$$g^C = (\partial g_{ij}) du^i du^j + 2g_{ij} du^i dq^j \tag{4.22}$$

i.e., the $(2n \times 2n)$ matrix of g^C is

$$g^C = \begin{pmatrix} \partial g_{ij} & g_{ij} \\ & \\ g_{ij} & 0 \end{pmatrix} \tag{4.23}$$

The following holds true:

Theorem 2 *Let (Q, g) be a Riemannian manifold. Then the system formed by the geodesic equation of g in Q and the Jacobi equation associated to g in TQ is the geodesic equation in TQ of the complete lift metric g^C. Therefore this system follows from a variational principle on TQ based on the energy functional defined by the lifted metric g^C.*

We see immediately that this Theorem is nothing but a simple consequence of our general results. In fact, the energy functional of g is based on the Lagrangian:

$$\mathcal{L} = \frac{1}{2} g_{ij} u^i u^j \tag{4.24}$$

whose associated first–order deformation Lagrangian is thence given by:

$$\mathcal{L}_1 = \frac{1}{2}(\partial_k g_{ij}) u^i u^j \eta^k + g_{ij} u^j \dot\eta^i \tag{4.25}$$

Using $\nabla(g) = 0$ this becomes immediately:

$$\mathcal{L}_1 = g_{ij}\left[\dot{\eta}^i + \Gamma^i_{mk} u^m \eta^k\right] u^j \tag{4.26}$$

Then $\mathcal{L}_1$ is in fact the energy Lagrangian of the lifted metric $g^C = 2g_{ij}\delta u^i dq^j$.

5. Conclusions

We have presented here a natural and direct generalisation of the Jacobi equation to the case of second–order Lagrangians, which are important not only for the sake of completeness but also and especially in view of applications to relativistic field theories (whereby gravitational Lagrangians depend effectively on second–order derivatives of a metric). Concrete applications to relativistic field theories will form the subject of forthcoming investigations.

Acknowledgements

One of us (B.C.) acknowledges support of GNSAGA-CNR and of MURST (Nat. Res. Proj. "Geometria delle Varietà Differenziabili").

One of us (M.F.) acknowledges support of GNFM-CNR and of MURST (Nat. Res. Proj. "Metodi Geometrici e Probabilistici in Fisica Matematica").

One of us (V.T.) has been partially supported by a special grant of Vicerrectoría Académica, Universidad de Concepción, by Dirección de Investigación, Universidad de Concepción, under contract 94.11.08-1, and by Visiting Professorships of the Italian GNFM-CNR and of the International Centre for Theoretical Physics, Trieste.

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