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A SUPERSYMMETRY APPROACH TO POISSON STRUCTURES OVER DIFFERENTIAL EQUATIONS

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ABSTRACT. An exposition of Poisson structures theory over nonlinear partial differential equations is given. The approach is based on consideration of d_h-invariant Hamiltonian formalism in the superalgebra $\Lambda^*(\mathcal{E}^\infty)$, d_h being the horizontal differential.

Relations between supersymmetries and Poisson structures are established. A local description of Poisson structures for the two cases is given: $\mathcal{E}^\infty = J^\infty(\pi)$ and $\mathcal{E}$ being a system of evolution equations.

1. Basic algebraic structures related to nonlinear partial differential equations.

When dealing with so-called integrable systems, one always meets with the following concepts:

- symmetries,
- conservation laws,
- Hamiltonian maps (Poisson structures),
- recursion operators.

In fact, all these concepts can be defined[1] *for any nonlinear partial differential equation* and only the problem *of existence* depend on a particular type of the equation at hand. Moreover, the basic constructions are of a purely algebraic (cohomological, to be more exact) nature and analytical reasons become important at the final step — in obtaining computational formulae.

Relations between basic PDE structures and algebraic constructions can be expressed by the following scheme:

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[1]and efficiently computed

	Conservation laws	Hamiltonian maps	Symmetries and recursion operators
Initial setting	Differential forms on $\mathcal{E}^\infty$, de Rham differential	$\mathcal{C}$-differential operators	Vector-valued forms on $\mathcal{E}^\infty$, U-differential
Defining structure	Cartan ideal	Schouten bracket	Nijenhuis bracket
Cohomology	Term E_1 of $\mathcal{C}$-spectral sequence	Homotopy classes	U-cohomology
Analytical formulae	$\ell_{\mathcal{E}}^*\varphi = 0$	$\Delta \circ \ell_{\mathcal{E}}^* + \ell_{\mathcal{E}} \circ \Delta = 0$	$\ell_{\mathcal{E}}\omega = 0$
References	[10, 12, 13, 15]	[1, 6, 7, 11]	[3, 4]

Consider these constructions in more details.

2. A general setting [8].

Let M be a smooth manifold, $\dim M = n$, $\pi : E \to M$ be a locally trivial vector bundle and $\pi_k : J^k(\pi) \to M$ be the bundle of it k-jets. We consider a formally integrable equation $\mathcal{E} \subset J^k(\pi)$ and its prolongations:

$$\mathcal{E}^s \subset J^{k+s}(\pi),\ \mathcal{E}^{s+1} \subset J^{k+s+1}(\pi), \dots, \mathcal{E}^\infty \subset J^\infty(\pi).$$

For any formally integrable equation $\mathcal{E}$ the Cartan connection

$$\mathcal{C} : D(M) \to D(\mathcal{E}^\infty),\ \mathcal{C} : X \mapsto \mathcal{C}X$$

is defined by

$$j_\infty^*(f)(\mathcal{C}X(\varphi)) = X(j_\infty^*(f)\varphi),\ \varphi \in C^\infty(\mathcal{E}^\infty), f \in \Gamma(\pi),$$

where for any manifold N by $D(N)$ denotes the Lie algebra of vector fields on N is denoted, $\Gamma(\pi)$ is the module of sections for π and $j_k(f)$ denotes the k-jet of the section f. The Cartan connection is flat, i.e.

$$\mathcal{C}[X, Y] = [\mathcal{C}X, \mathcal{C}Y],\ X, Y \in D(M)$$

and in local coordinate presentation coincides with the total derivatives.

The Cartan connection determines two mutually dual splittings

$$D(\mathcal{E}^\infty) = \mathcal{C}D(\mathcal{E}^\infty) \oplus D^v(\mathcal{E}^\infty)$$

and

$$\Lambda^1(\mathcal{E}^\infty) = \mathcal{C}^1\Lambda(\mathcal{E}^\infty) \oplus \Lambda^1_h(\mathcal{E}^\infty), \tag{1}$$

where $\mathcal{C}D(\mathcal{E}^\infty)$ is spanned in $D(\mathcal{E}^\infty)$ by the fields of the form $\mathcal{C}X$, $X \in D(M)$, $D^v(\mathcal{E}^\infty)$ consists of π_∞-vertical fields, $\mathcal{C}^1\Lambda(\mathcal{E}^\infty)$ are 1-forms vanishing on $\mathcal{C}D(\mathcal{E}^\infty)$ and $\Lambda^1_h(\mathcal{E}^\infty)$ are horizontal 1-forms on $\mathcal{E}^\infty$. From (1) it follows that

$$\Lambda^i(\mathcal{E}^\infty) = \sum_{p+q=i} \mathcal{C}^p\Lambda(\mathcal{E}^\infty) \wedge \Lambda^q_h(\mathcal{E}^\infty), \tag{2}$$

where

$$\mathcal{C}^p\Lambda(\mathcal{E}^\infty) = \underbrace{\mathcal{C}^1\Lambda(\mathcal{E}^\infty) \wedge \ldots \wedge \mathcal{C}^1\Lambda(\mathcal{E}^\infty)}_{p \text{ times}},$$

while $\Lambda^q_h(\mathcal{E}^\infty)$ is the module of horizontal q-forms.

3. The $\mathcal{C}$-spectral sequence [10, 12, 13, 15].

Consider the ideal $\mathcal{C}\Lambda^*(\mathcal{E}^\infty)$ generated in $\Lambda^*(\mathcal{E}^\infty)$ by $\mathcal{C}^1\Lambda(\mathcal{E}^\infty)$. Then the filtration in $\Lambda^*(\mathcal{E}^\infty)$ by the powers of $\mathcal{C}\Lambda^*(\mathcal{E}^\infty)$ is in agreement with the action of the de Rham differential d. Corresponding spectral sequence (so-called Vinogradov's $\mathcal{C}$-spectral sequence) converges to de Rham cohomology of M and possesses the following properties[2]:

Theorem 1. Let $\mathcal{E} \subset J^k(\pi)$ be an equation. Then locally one has:

(i) The term $E_0 = E_0(\mathcal{E})$ is of the form $E_0^{p,q} = \mathcal{C}^p\Lambda(\mathcal{E}^\infty) \wedge \Lambda^q_h(\mathcal{E}^\infty)$ while $d_0^{p,q}$ coincides with the horizontal differential d_h with coefficients in $\mathcal{C}^p\Lambda(\mathcal{E}^\infty)$.

(ii) The group $E_1^{0,n-1}$ coincides with the group of conservation laws of the equation $\mathcal{E}$.

(iii) The group $E_1^{1,n-1}$ consists of generating functions of conservation laws.

(iv) In the case $\mathcal{E}^\infty = J^\infty(\pi)$, only the groups $E_1^{0,0}$ and $E_1^{p,n}$ are nontrivial in the term E_1, $p \geq 0$.

(v) If $\mathcal{E}$ is a "good" (roughly speaking, not highly overdetermined) equation, only the groups $E_1^{0,0}$, $E_1^{p,n-1}$ and $E_1^{p,n}$ are nontrivial in the term E_1, $p \geq 0$. In this case the differential $d_1^{0,n-1} : E_1^{p,n-1} \to E_1^{p,n}$ is an embedding.

(vi) Generating functions of conservation laws satisfy the equation

$$\ell^*_{\mathcal{E}}\varphi = 0, \tag{3}$$

where $\ell^*_{\mathcal{E}}$ is the formally adjoint to the universal linearization operator of the equation $\mathcal{E}$.

[2]To simplify exposition, we consider the case $H^*(M, \mathbb{R}) = \mathbb{R}$.

4. The U-cohomology [3, 4].

This cohomology theory arises via the following scheme. Let $\boldsymbol{k}$ be a zero characteristic field and A be a commutative unitary $\boldsymbol{k}$-algebra. Consider tensor modules $\Lambda^i(A)\otimes_A D(A)$, where $D(A)$ is the Lie algebra of derivations $A\to A$ and define a contraction

$$(\omega\otimes X)\rfloor\theta \overset{\text{def}}{=} \omega\wedge(X\rfloor\theta),\ \ \omega,\theta\in\Lambda^*(A), X\in D(A).$$

Then, using the standard scheme [2] one defines the operation of "Lievization" $L_\Omega : \Lambda^*(A)\to\Lambda^*(A)$, $\Omega\in\Lambda^i(A)\otimes_A D(A)$, and it can be shown that the commutator $[L_\Omega, L_\Theta]$ is again of the form L_Ξ for some $\Xi\in\Lambda^*(A)\otimes_A D(A)$. The element $\Xi\overset{\text{def}}{=}[\![\Omega,\Theta]\!]^{FN}$ is uniquely defined and is called the Frölicher-Nijenhuis bracket of Ω and Θ. The module $\Lambda^*(A)\otimes D(A)$ is a super Lie algebra with respect to $[\![\bullet,\bullet]\!]^{FN}$ and for any element $\Omega\in\Lambda^1(A)\otimes D(A)$ the map $\partial_\Omega\overset{\text{def}}{=}[\![\Omega,\bullet]\!]^{FN}$ satisfies $\partial_\Omega\circ\partial_\Omega=0$.

Let $U_{\mathcal{C}}\in\Lambda^1(\mathcal{E}^\infty)\otimes D^v(\mathcal{E}^\infty)$ be the connection form of the Cartan connection on $\mathcal{E}$. Then flatness of $\mathcal{C}$ is equivalent to $[\![U_{\mathcal{C}},U_{\mathcal{C}}]\!]^{FN}=0$. Corresponding differential can be restricted onto $\Lambda^*(\mathcal{E}^\infty)\otimes D^v(\mathcal{E}^\infty)$ and this restriction is denoted by $\partial_{\mathcal{C}}$. The groups $H^*_{\mathcal{C}}(\mathcal{E}^\infty)\overset{\text{def}}{=}\ker\,\partial_{\mathcal{C}}/\operatorname{im}\,\partial_{\mathcal{C}}$ are called U-cohomology groups of the equation $\mathcal{E}$ and enjoy the following properties:

Theorem 2. Let $\mathcal{E}\subset J^k(\pi)$ be a formally integrable equation. Then (i) Consider splitting (2). Then

$$\partial_{\mathcal{C}}(\mathcal{C}^p(\mathcal{E}^\infty)\wedge\Lambda^q_h(\mathcal{E}^\infty)\otimes D^v(\mathcal{E}^\infty))\subset\mathcal{C}^p(\mathcal{E}^\infty)\wedge\Lambda^{q+1}_h(\mathcal{E}^\infty)\otimes D^v(\mathcal{E}^\infty).$$

(ii) The differential

$$\partial^{p,q}_{\mathcal{C}}:\mathcal{C}^p(\mathcal{E}^\infty)\wedge\Lambda^q_h(\mathcal{E}^\infty)\otimes D^v(\mathcal{E}^\infty)\to\mathcal{C}^p(\mathcal{E}^\infty)\wedge\Lambda^{q+1}_h(\mathcal{E}^\infty)\otimes D^v(\mathcal{E}^\infty)$$

coincides with the horizontal differential with the coefficients in $\mathcal{C}^p(\mathcal{E}^\infty)\otimes D^v(\mathcal{E}^\infty)$. Consequently, one has the splitting $H^i_{\mathcal{C}}(\mathcal{E}^\infty)=\sum_{p+q=i}H^{p,q}_{\mathcal{C}}(\mathcal{E}^\infty)$.

(iii) The group $H^*_{\mathcal{C}}(\mathcal{E}^\infty)$ is a module over the term E_1 of the $\mathcal{C}$-spectral sequence and inherits both the contraction operation and the Frölicher-Nijenhuis bracket from $\Lambda^*(\mathcal{E}^\infty)\otimes D^v(\mathcal{E}^\infty)$.

(iv) For any formally integrable equation $\mathcal{E}$ one has

$$H^{p,0}_{\mathcal{C}}=\ker\,\ell^{[p]}_{\mathcal{E}},$$

where $\ell^{[p]}_{\mathcal{E}}:\mathcal{C}^p\Lambda(\mathcal{E}^\infty)\to\mathcal{C}^p\Lambda(\mathcal{E}^\infty)$ is a natural extension of the universal linearization onto $\mathcal{C}^p\Lambda(\mathcal{E}^\infty)$. The correspondence between elements of $\ker\,\ell^{[p]}_{\mathcal{E}}$ is established by the formula

$$\Omega\mapsto\ni_\Omega\overset{\text{def}}{=}\sum D_\sigma(\Omega)\frac{\partial}{\partial u^\alpha_\sigma},$$

where u^α_σ in $J^\infty(\pi)$ are local coordinates corresponding to local coordinates $(x_1,\dots,x_n,u^1,\dots,u^m)$ in $J^0(\pi)$ and $D_\sigma\overset{\text{def}}{=}D^{\sigma_1}_{x_1}\circ\dots\circ D^{\sigma_n}_{x_n}$ for $\sigma=(\sigma_1,\dots,\sigma_n)$ and D_{x_i} being the total derivative along x_i.

(v) In particular, $H^{0,0}_{\mathcal{C}} = \ker \ell_{\mathcal{E}}$ coincides with the Lie algebra $\operatorname{sym}\mathcal{E}$ of higher symmetries for $\mathcal{E}$.

(vi) The group $H^{1,0}_{\mathcal{C}}$ is an associative algebra with respect to contraction operation $\rfloor$ and $U_{\mathcal{C}}$ is its unit.

(vii) The action $H^{0,0}_{\mathcal{C}} \rfloor H^{1,0}_{\mathcal{C}} \subset H^{0,0}_{\mathcal{C}}$ is a representation of this algebra and the elements of the algebra $H^{1,0}_{\mathcal{C}} = \ker \ell^{[1]}_{\mathcal{E}}$ can be identified with recursion operators for symmetries of $\mathcal{E}$. To be more precise, if $\omega \in \mathcal{C}^1\Lambda(\mathcal{E}^\infty)$ is a solution of the equation

$$\ell^{[1]}_{\mathcal{E}}\omega = 0 \tag{4}$$

and $\varphi \in \operatorname{sym}\mathcal{E}$ is a symmetry, then the action of corresponding recursion operator is expressed by

$$\mathcal{R}_\omega\varphi \stackrel{\text{def}}{=} \ni_\varphi \rfloor \omega.$$

5. Super Poisson structures [5] and δ-invariant theory.

Let again A be a commutative unitary $\boldsymbol{k}$-algebra and $\mathcal{A} = \sum_{\alpha\geq 0}\mathcal{A}_\alpha$, $\mathcal{A}_0 = A$, be a graded commutative A-algebra, i.e.

$$ab = (-1)^{ab}ba$$

for any $a, b \in \mathcal{A}$ (here and below, to simplify the notations, we write $(-1)^\theta$ instead of $(-1)^{\deg\theta}$. For any $i \geq 0$ the module $D_i(\mathcal{A})$ of i-derivations is defined together with the evaluation map

$$D_i(\mathcal{A}) \otimes_{\boldsymbol{k}} \mathcal{A} \to D_{i-1}(\mathcal{A}),\ \Delta \mapsto \Delta(a)\ \Delta \in D_i(\mathcal{A}), a \in \mathcal{A}.$$

For any two elements $\Delta \in D_i(\mathcal{A}), \Delta' \in D_j(\mathcal{A})$ we define by induction their Schouten bracket $[\![\Delta, \Delta']\!]^S \in D_{i+j-1}(\mathcal{A})$ by setting $[\![\Delta, \Delta']\!]^S = [\Delta, \Delta']$, when $\Delta, \Delta' \in D_1(\mathcal{A})$, and

$$[\![\Delta, \Delta']\!]^S a = [\![\Delta, \Delta' a]\!]^S + (-1)^{a\Delta'+j}[\![\Delta a, \Delta']\!]^S,\ a \in \mathcal{A}.$$

Then $D_*(\mathcal{A}) = \sum_{i\geq 0} D_i(\mathcal{A})$ is a bigraded Lie algebra with respect to $[\![\bullet, \bullet]\!]^S$.

We say that a bivector $P \in D_2(\mathcal{A})$ is a super Poisson structure on $\mathcal{A}$, if (a) P is an even element and (b)

$$[\![P, P]\!]^S = 0. \tag{5}$$

We also say that two Poisson structures P, P' commute, if $[\![P, P']\!]^S = 0$. To any Poisson structure P we relate corresponding Poisson map $f_P : \Lambda^1(\mathcal{A}) \to D_1(\mathcal{A})$ defined by $f_P(da) = Pa$, d being the de Rham differential.

For any $a, b \in \mathcal{A}$ one defines the Poisson bracket corresponding to P by $\{a, b\}_P \stackrel{\text{def}}{=} P(a, b)$. The basic property (5) of P implies that this bracket satisfies the super Jacobi identity.

Assume now that the algebra $\mathcal{A}$ is endowed with an odd differential $\delta : \mathcal{A} \to \mathcal{A}$, $\delta \circ \delta = 0$. Then δ-invariant calculus over $\mathcal{A}$ can be constructed. Namely, we consider differential operators $\Delta : \mathcal{A} \to \mathcal{A}$ satisfying $[\delta, \Delta] = 0$ and classes of such operators modulo operators of the form $\Delta = [\delta, \nabla]$. Denote the set of these classes by $\mathrm{Diff}(\mathcal{A};\delta)$. Then $\mathrm{Diff}(\mathcal{A};\delta)$ acts on the homology group $H(\mathcal{A};\delta) = \ker\delta/\operatorname{im}\delta$ and in fact it is this action which really matters in the theory.

The set $\mathrm{Diff}(\mathcal{A};\delta)$ inherits the associative algebra structure with respect to composition of differential operators. One can extend the action of δ onto $\mathrm{Diff}(\mathcal{A};\delta)$ by setting $\delta([\Delta]) = [\delta\circ\Delta]$, where $\Delta : \mathcal{A} \to \mathcal{A}$ is a differential operator and $[\cdot]$ denotes its class. Obviously, $\delta\circ\delta = 0$ again which makes it possible to define modules $\mathrm{Diff}^i(\mathcal{A};\delta) \overset{\text{def}}{=} \mathrm{Diff}(\mathcal{A}, \mathrm{Diff}^{i-1}(\mathcal{A};\delta);\delta)$. Thus, the whole conceptual scheme of differential calculus in commutative algebras [14] can be carried over into the δ-invariant setting. In particular, one can define modules of δ-invariant polyderivations $D_i(\mathcal{A};\delta)$ as well as of symmetric $\mathrm{Diff}_i^{\mathrm{sym}}(\mathcal{A};\delta)$ and skew-symmetric $\mathrm{Diff}_i^{\mathrm{alt}}(\mathcal{A};\delta)$ polydifferential operators respectively and define the evaluation maps

$$\begin{aligned}
D_i(\mathcal{A};\delta)\otimes H(\mathcal{A};\delta) &\to D_{i-1}(\mathcal{A};\delta),\\
\mathrm{Diff}_i^{\mathrm{sym}}(\mathcal{A};\delta)\otimes H(\mathcal{A};\delta) &\to \mathrm{Diff}_{i-1}^{\mathrm{sym}}(\mathcal{A};\delta),\\
\mathrm{Diff}_i^{\mathrm{alt}}(\mathcal{A};\delta)\otimes H(\mathcal{A};\delta) &\to \mathrm{Diff}_{i-1}^{\mathrm{alt}}(\mathcal{A};\delta).
\end{aligned}$$

The construction of super Schouten bracket is obviously δ-invariant and consequently one obtains the brackets

$$\begin{aligned}
[\![D_i(\mathcal{A};\delta), D_j(\mathcal{A};\delta)]\!]^S &\subset D_{i+j}(\mathcal{A};\delta),\\
[\![\,\mathrm{Diff}_i^{\mathrm{alt}}(\mathcal{A};\delta), \mathrm{Diff}_j^{\mathrm{alt}}(\mathcal{A};\delta)]\!]^S &\subset \mathrm{Diff}_{i+j}^{\mathrm{alt}}(\mathcal{A};\delta).
\end{aligned}$$

A δ-invariant Poisson structure in $\mathcal{A}$ is defined as an even element $\mathcal{P}\in D_2(\mathcal{A};\delta)$ satisfying $[\![\mathcal{P},\mathcal{P}]\!]^S = 0$. In a similar manner one can define higher Poisson structures as elements of $\mathrm{Diff}_2^{\mathrm{alt}}(\ CA;\delta)$ complying with the same conditions.

6. Realization for nonlinear differential equations.

Let now $\mathcal{E}$ be a formally integrable equation and take the algebra $\Lambda^*(\mathcal{E}^\infty) = \sum_{p,q}\mathcal{C}^p\Lambda(\mathcal{E}^\infty)\wedge\Lambda_h^q\Lambda(\mathcal{E}^\infty)$ for the $\mathcal{A}$ and the differential $d_0 : \mathcal{C}^p\Lambda(\mathcal{E}^\infty)\wedge\Lambda_h^q(\mathcal{E}^\infty) \to \mathcal{C}^p\Lambda(\mathcal{E}^\infty)\wedge\Lambda_h^{q+1}(\mathcal{E}^\infty)$ of the of $\mathcal{C}$-spectral sequence term E_0 for δ. Use the notation $D_i(\Lambda^*(\mathcal{E}^\infty);d_0) \overset{\text{def}}{=} \mathcal{D}_i(\mathcal{E})$.

Let us first describe the group $\mathcal{D}_1(\mathcal{E})$. To do this, we use the following well-known property of graded derivations of the algebra $\Lambda^*(N)$, N being a smooth manifold: Any $X\in D_1(\Lambda^*(N))$ is uniquely represented as the sum of a Lie derivative and a contraction:

$$X = L_{X_0} + i_{X_1}, \tag{6}$$

for some uniquely determined $X_0, X_1 \in \Lambda^*(N)\otimes D(N)$. Thus we have the maps

$$\begin{aligned}
\Delta_0 &: D(\Lambda^*(N)) \to \Lambda^*(N)\otimes D(N),\ \Delta_0(X) = X_0,\\
\Delta_1 &: D(\Lambda^*(N)) \to \Lambda^*(N)\otimes D(N),\ \Delta_1(X) = X_1,
\end{aligned}$$

the first one being a homomorphism and the second one being a first order differential operator.

Using decomposition (6) and basic properties of the Frölicher-Nijenhuis bracket [4], one can see that

$$
\begin{aligned}
&[X,Y]_0 = [\![X_0,Y_0]\!]^{FN} + X_1 \rfloor Y_0 - (-1)^{XY} Y_1 \rfloor X_0,\\
(7)\qquad &[X,Y]_1 = [\![X_1,Y_1]\!]^{RN} + [\![X_0,Y_1]\!]^{FN} - (-1)^{XY} [\![Y_0,X_1]\!]^{FN},
\end{aligned}
$$

for any $X, Y \in D(\Lambda^*(N))$. Here $[\![\bullet,\bullet]\!]^{RN}$ is the so-called Richardson-Nijenhuis bracket defined as

$$[\![\Omega,\Theta]\!]^{RN} \stackrel{\text{def}}{=} \Omega \rfloor \Theta - (-1)^{(\Omega+1)(\Theta+1)} \Theta \rfloor \Omega,$$

for any $\Omega, \Theta \in \Lambda^*(N) \otimes D(N)$.

The following description of $\mathcal{D}_1(\mathcal{E})$ is valid:

Theorem 3. For any formally integrable equation $\mathcal{E}$ one has an isomorphism

$$\mathcal{D}_1(\mathcal{E}) = H^*_{\mathcal{C}}(\mathcal{E}^\infty) \oplus H^*_{\mathcal{C}}(\mathcal{E}^\infty).$$

In particular, the action of elements $\mathcal{X} = (X_0, X_1) \in H^{*,0}_{\mathcal{C}}(\mathcal{E}^\infty) \oplus H^{*,0}_{\mathcal{C}}(\mathcal{E}^\infty)$ on $\mathcal{D}_0(\mathcal{E}) = E_1^{*,*}$ is as follows:

$$([\Omega_0],[\Omega_1])[\omega] = [L_{\ni_{\Omega_0}} \omega + i_{\ni_{\Omega_1}} \omega],$$

where $[\cdot]$ denotes the cohomology class.

We use the notation $\mathcal{S}(\mathcal{E}) = H^{*,0}_{\mathcal{C}}(\mathcal{E}^\infty) \oplus H^{*,0}_{\mathcal{C}}(\mathcal{E}^\infty)$ and call elements of $\mathcal{S}(\mathcal{E})$ *supersymmetries of the equation* $\mathcal{E}$. Note that the commutator of superderivations induces a bracket in $\mathcal{S}(\mathcal{E})$ which combines elements of the Jacobi-Richardson-Frölicher-Nijenhuis brackets, namely, if $\mathcal{X} = (\ni_{\Omega_0}, \ni_{\Omega_1})$ and $\mathcal{Y} = (\ni_{\Theta_0}, \ni_{\Theta_1})$, then $[\mathcal{X},\mathcal{Y}] = (\ni_{\Xi_0}, \ni_{\Xi_1})$ with

$$
\begin{aligned}
\Xi_0 &= \{\Omega_0,\Theta_0\} + \ni_{\Omega_0} \rfloor \Omega_1 - (-1)^{\Omega_0(\Theta_1+1)} \ni_{\Omega_0} \rfloor \Theta_1,\\
\Xi_1 &= \ni_{\Omega_1} \rfloor \Theta_1 - (-1)^{(\Omega_1+1)(\Theta_1+1)} \ni_{\Theta_1} \rfloor \Omega_1 +\\
&\qquad \{\Omega_0,\Theta_1\} - (-1)^{\Theta_0(\Omega_1+1)} \{\Theta_0,\Omega_1\},
\end{aligned}
$$

where $\{\bullet,\bullet\}$ denotes the super Jacobi bracket.

We proceed now with description of $\mathcal{D}_2(\mathcal{E})$. Note that having an element $X \in D_2(\Lambda^*(\mathcal{E}^\infty))$ we get the evaluation map $f_X : \Lambda^*(\mathcal{E}^\infty) \to D_1(\mathcal{E}^\infty)$ determined by $f_X(\omega) = X\omega$. Taking compositions $f_X \circ \Delta_i$, $i = 0, 1$, we obtain the operators

$$
\begin{aligned}
X_0 &: \Lambda^*(\mathcal{E}^\infty) \to \Lambda^*(\mathcal{E}^\infty) \otimes D(\mathcal{E}^\infty),\\
X_1 &: \Lambda^*(\mathcal{E}^\infty) \to \Lambda^*(\mathcal{E}^\infty) \otimes D(\mathcal{E}^\infty).
\end{aligned}
$$

It can be shown [6] that when one considers d_0-homotopy classes, then description of these classes reduces to description of classes of second order differential operators $\Delta : \Lambda^*(\mathcal{E}^\infty) \to \Lambda^*(\mathcal{E}^\infty) \otimes D^v(\mathcal{E}^\infty)$ satisfying

$$\Delta \circ d_0 + \partial_{\mathcal{C}} \circ \Delta = 0. \tag{8}$$

To be more precise, any element of $\mathcal{D}_2(\mathcal{E})$ is determined by a pair (Δ_0, Δ_1) of differential operators satisfying (8), such that Δ_0 is a derivation and Δ_1 is a second order differential operator whose symbol is determined by Δ_0 (the details see in [6]. Due to the two-lines theorem [15] one obtains that for "good" equations such operators act trivially on corresponding cohomologies.

This observation forces one to generalize the concept of a Poisson structure for differential equations. We do it in the next section.

7. Poisson structures and superization.

Definition 1. Let $\mathcal{E} \subset J^k(\pi)$ be a formally integrable equation.

(i) Say that a pair $P = (\Delta_0, \Delta_1)$ of $\mathcal{C}$-differential operators

$$\Delta_0, \Delta_1 : \Lambda^*(\mathcal{E}^\infty) \to \Lambda^*(\mathcal{E}^\infty) \otimes D^v(\mathcal{E}^\infty)$$

is Poissonnian, if

(a) both Δ_0 and Δ_1 satisfy equation (8), Δ_0 is an even operator and $\deg \Delta_1 = \deg \Delta_0 + 1$;

(b) The bracket $\{\bullet, \bullet\}_P$ defined by

$$\{\omega, \theta\}_P \stackrel{\text{def}}{=} L_{\Delta_0 \omega}\theta + i_{\Delta_1 \omega}\theta$$

is skew-symmetric and satisfies the graded Jacobi identity.

(ii) A Poisson structure on $\mathcal{E}$ related to the pair $P = (\Delta_0, \Delta_1)$ is the corresponding cohomology map

$$\mathcal{P} : E_1^{*,*} \to H_{\mathcal{C}}^*(\mathcal{E}^\infty) \oplus H_{\mathcal{C}}^*(\mathcal{E}^\infty).$$

(iii) A Poisson structure is said to be of a normal type, if

$$\mathcal{P}(E_1^{*,p_0}) \subset H_{\mathcal{C}}^{*,0}(\mathcal{E}^\infty) \oplus H_{\mathcal{C}}^{*,0}(\mathcal{E}^\infty),$$

where p_0 is the number of the first nontrivial line in $E_1(\mathcal{E})$.

Thus, Poisson structures of a normal type send conservation laws of the equation $\mathcal{E}$ (as well as other elements of the $(n-1)$-st line in E_1) to supersymmetries of this equation.

To get a characterization of the above defined Poisson structures we need the concept of *superization* of the equation $\mathcal{E}$ (see [3]).

Namely, extend the algebra $C^\infty(\mathcal{E}^\infty)$ by adding Cartan forms to it, i.e. elements of the modules $\mathcal{C}^p\Lambda(\mathcal{E}^\infty)$, $p > 0$, as variables of the degree p. Then the Cartan connection extends onto $\mathcal{C}^*\Lambda(\mathcal{E}^\infty) = \sum_{p\geq 0} \mathcal{C}^p\Lambda(\mathcal{E}^\infty)$. In fact

$$L_{\mathcal{C}(fX)}\omega = L_{f\mathcal{C}X}\omega = fL_{\mathcal{C}X}\omega + d\omega \wedge \mathcal{C}X \rfloor \omega = fL_{\mathcal{C}X}\omega.$$

Hence, one can introduce an object whose function algebra is generated over $C^\infty(\mathcal{E}^\infty)$ by new *odd* variables $\omega \in \mathcal{C}^1\Lambda(\mathcal{E}^\infty)$. This object can be interpreted as *a super differential equation.* Denote it by $\mathcal{S}(\mathcal{E})$. Consequently, $C^\infty(\mathcal{S}\mathcal{E}) = \sum_{p\geq 0} \mathcal{C}^p)\Lambda(\mathcal{E}^\infty)$. Then all basic concepts, such as symmetries, conservation laws, etc., can be generalized onto this object. In particular, one has

Proposition 4. For any formally integrable equation $\mathcal{E}$ the following facts hold:

(i) Symmetries of superization coincide with supersymmetries of the initial equation, i.e.

$$\mathrm{sym}\,(\mathcal{S}\mathcal{E}) = \mathcal{S}(\mathcal{E}).$$

(ii) The system $(E_0^{p,q}(\mathcal{E}), d_0)$ is equivalent to the horizontal complex of the superized equation $\mathcal{S}(\mathcal{E})$. In particular, the group of conservation laws for the superized equation coincides with $E_1^{*,n-1}(\mathcal{E})$.

(iii) The system $(\mathcal{C}^p\Lambda(\mathcal{E}^\infty)\wedge\Lambda_h^q(\mathcal{E}^\infty)\otimes D^v(\mathcal{E}^\infty), \partial_{\mathcal{C}})$ for the equation $\mathcal{E}$ is equivalent to the 0-th column of a similar system for superization.

The identifications above allow one to prove the following result [6]:

Theorem 5. Let $\mathcal{E}$ be the "empty" equation, i.e. $\mathcal{E}^\infty = J^\infty(\pi)$. Then locally one has

(i) All nontrivial Poisson structures for $\mathcal{E}$ are of a normal type.

(ii) To any Poisson structure $\mathcal{P}$ on $\mathcal{E}$ there corresponds a super $\mathcal{C}$-differential operator $\Delta : E_1^{1,n}(\mathcal{S}\mathcal{E}) \to \mathrm{sym}\,(\mathcal{S}\mathcal{E})$ by which $\mathcal{P}$ is determines uniquely.

(iii) Corresponding Poisson bracket $\{\bullet,\bullet\}_\Delta$ if skew-symmetric if and only if

$$\Delta + \Delta^* = 0, \tag{9}$$

where Δ^* is formally adjoint to Δ.

(iv) The bracket $\{\bullet,\bullet\}_\Delta$ satisfies the graded Jacobi identity if and only if

$$[\ni_{\Delta\varphi}, \Delta] = \ell_{\Delta\varphi}\circ\Delta + \Delta\circ\ell^*_{\Delta\varphi}, \tag{10}$$

where $\ni_\psi$ is the evolutionary super vector field corresponding to ψ.

(v) Two Poisson structures $\mathcal{P}$ and $\mathcal{P}'$ commute if and only if

$$[\ni_{\Delta\varphi}, \Delta'] + [\ni_{\Delta'\varphi}, \Delta] = \ell_{\Delta\varphi}\circ\Delta' + \ell_{\Delta'\varphi}\circ\Delta + \Delta\circ\ell^*_{\Delta'\varphi} + \Delta'\circ\ell^*_{\Delta\varphi}. \tag{11}$$

The proof of this theorem uses the theory of formally adjoint operator presented for the graded case in [9] and is a straightforward generalizations of the results obtained in [1, 11].

Previous result easily generalizes to the case of a determined system of evolution equations [7] and we obtain the following

Theorem 6. Let $\mathcal{E}$ be a determined system of evolution equations presented as

$$u_t^j = \Psi^j(x, \ldots, u_\sigma^\alpha, \ldots),$$

where $j, \alpha = 1, \ldots, m,\ |\sigma| \le k,\ x = (x_1, \ldots, x_n)$ and

$$u_\sigma^j = \frac{\partial^{|\sigma|}u^j}{\partial x_1^{\sigma_1}\ldots\partial x_n^{\sigma_n}}.$$

Then Poisson structures of a normal type on $\mathcal{E}$ are determined by operators Δ satisfying Theorem 5 and such that

$$\ell_{\mathcal{E}}\circ\Delta + \Delta\circ\ell^*_{\mathcal{E}} = 0, \tag{12}$$

where $\ell_{\mathcal{E}}$ is the universal linearization operator for the equation $\mathcal{E}$ and $\ell^*_{\mathcal{E}}$ is its formally adjoint.

8. Example.

Consider the KdV equation $\mathcal{E}$ presented as

$$u_t = uu_x + u_{xxx}.$$

Then solving (12) in this case one obtains two independent solutions

$$\Delta_1 = D_x,$$

where D_x is the total derivative with respect to x, and

$$\Delta_2 = D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_x,$$

which coincide with well-known Hamiltonian maps for the KdV.

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