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INFINITE DIMENSIONAL LIE ALGEBRA COHOMOLOGY AND THE COHOMOLOGY OF INVARIANT EULER–LAGRANGE COMPLEXES: A PRELIMINARY REPORT

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ABSTRACT. Let $\pi\colon E \to M$ be a fiber bundle and let G be an infinite dimensional Lie pseudo-group of projectable transformations acting on E. We outline a general method by which the local cohomology of the G invariant Euler-Lagrange complex on E can be computed in terms of the Lie algebra cohomology of the formal infinitesimal generators of G. Our method is illustrated with two examples.

§1. Introduction. Let $\pi\colon E \to M$ be a fiber bundle and let G be a finite or infinite dimensional Lie pseudo-group of π projectable transformations on E. In this expository article we shall describe a general method for computing the cohomology of the G invariant Euler-Lagrange complex for E in terms of the continuous Lie algebra cohomology of the formal infinitesimal generators of G. We shall illustrate our results with 2 specific examples. The first example finds its origins in the celebrated imbedding theorems of Whitney as applied to immersed curves in $\mathbf{R}^2$ and $\mathbf{R}^3$. The second example is motivated by the Chern-Weil homomorphism in the simple case of $U(1)$ connections and by applications to gauge invariant vector field theories. We shall analyze these two examples in some detail, not so much perhaps because of their intrinsic merit, but more because they underscore the specific computations needed to support our general theoretical framework. Our examples make apparent, and for us this is the most salient point of the present article, the central role that infinite dimensional Lie algebra cohomology plays in certain classification problems in differential geometry and mathematical physics.

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Detailed proofs of our general method and a wider variety of examples and applications will be presented elsewhere.

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§2. Invariant Variational Bicomplexes. We are given a fiber bundle $\pi\colon E \to M$ and a finite dimensional Lie transformation group or an infinite dimensional Lie pseudo-group of projectable transformations G on E. We let Γ be the Lie algebra of vector fields on E determined by the infinitesimal action of G on E. To define the G invariant Euler-Lagrange complex for E, we first construct the associated variational bicomplex. Let $\pi^\infty_M\colon J^\infty(E) \to M$ be the infinite jet bundle of E and let

$$\Omega^p(J^\infty(E)) = \bigoplus_{r+s=p} \Omega^{r,s}(J^\infty(E)) \tag{2.1}$$

be the decomposition of the p forms on $J^\infty(E)$ into forms of horizontal degree r and vertical degree s. By definition, $\omega \in \Omega^{r,s}(J^\infty(E))$ if and only if

$$\omega(X_1, X_2, \dots, X_p) = 0$$

whenever more than s of the vectors X_1, X_2, ... , X_p are π^∞_M vertical or more than r of the vectors X_1, X_2, ... , X_p are total vectors. The projection maps associated to the direct sum decomposition (2.1) are denoted by

$$\pi^{r,s}\colon \Omega^p(J^\infty(E)) \to \Omega^{r,s}(J^\infty(E)). \tag{2.2}$$

The *augmented variational bicomplex on* $J^\infty(E)$ is the double complex

$$\begin{tikzcd}
 & & {} & {} & & {} & {} & {} & \\
 & 0 \arrow[r] & \Omega^{0,3} \arrow[u, "d_V"'] & & \cdots & & \Omega^{n,3} \arrow[u, "d_V"'] \arrow[r, "I"] & \mathcal{F}^3 \arrow[u, "\delta_V"'] \arrow[r] & 0 \\
 & 0 \arrow[r] & \Omega^{0,2} \arrow[u, "d_V"'] \arrow[r, "d_H"] & \Omega^{1,2} \arrow[r, "d_H"] & \cdots & \Omega^{n-1,2} \arrow[r, "d_H"] & \Omega^{n,2} \arrow[u, "d_V"'] \arrow[r, "I"] & \mathcal{F}^2 \arrow[u, "\delta_V"'] \arrow[r] & 0 \\
 & 0 \arrow[r] & \Omega^{0,1} \arrow[u, "d_V"'] \arrow[r, "d_H"] & \Omega^{1,1} \arrow[u, "d_V"'] \arrow[r, "d_H"] & \cdots & \Omega^{n-1,1} \arrow[u, "d_V"'] \arrow[r, "d_H"] & \Omega^{n,1} \arrow[u, "d_V"'] \arrow[r, "I"] & \mathcal{F}^1 \arrow[u, "\delta_V"'] \arrow[r] & 0 \\
0 \arrow[r] & \mathbf{R} \arrow[r] & \Omega^{0,0} \arrow[u, "d_V"'] \arrow[r, "d_H"] & \Omega^{1,0} \arrow[u, "d_V"'] \arrow[r, "d_H"] & \cdots & \Omega^{n-1,0} \arrow[u, "d_V"'] \arrow[r, "d_H"] & \Omega^{n,0} \arrow[u, "d_V"'] & &
\end{tikzcd} \tag{2.2}$$

The exterior derivative d on $J^\infty(E)$ decomposes as $d = d_H + d_V$; the linear, total differential operator I is the integration by parts operator

$$I\colon \Omega^{n,s}(J^\infty(E)) \to \Omega^{n,s}(J^\infty(E))$$

as characterized by the properties

$$I \circ d_H = 0, \qquad I^2 = I, \qquad d_V \circ I = I \circ d_V;$$

the spaces $\mathcal{F}^s(J^\infty(E))$ of functional s forms are defined by

$$\mathcal{F}^s(J^\infty(E)) = \{\, \omega \in \Omega^{n,s}(J^\infty(E)) \mid I(\omega) = \omega \,\} = \operatorname{im}(I);$$

and the differential δ_V is the induced map

$$\delta_V = I \circ d_V.$$

At the end of this section we give a local coordinate description of the augmented variational bicomplex. For further details of this construction, see [1], [2], [9], [11].

The edge complex of the augmented variational bicomplex is the Euler- Lagrange complex, namely

$$\begin{aligned} 0 \longrightarrow \mathbf{R} \longrightarrow \Omega^{0,0} \xrightarrow{d_H} \Omega^{1,0} \xrightarrow{d_H} \cdots \\ \xrightarrow{d_H} \Omega^{n-1,0} \xrightarrow{d_H} \Omega^{n,0} \xrightarrow{\delta_V} \mathcal{F}^1 \xrightarrow{\delta_V} \mathcal{F}^2 \xrightarrow{\delta_V} \mathcal{F}^3 \xrightarrow{\delta_V} \cdots . \end{aligned} \tag{2.3}$$

The differential $\delta_V : \Omega^{n,0}(J^\infty(E)) \to \mathcal{F}^1(J^\infty(E))$ is precisely the classical Euler-Lagrange operator from the calculus of variations and the differential $\delta_V : \mathcal{F}^1 \to \mathcal{F}^2$ is the Helmholtz operator which arises in the inverse problem of the calculus of variations.

We define the projection map $\Pi\colon \Omega^p(J^\infty(E)) \to \mathcal{E}^p(J^\infty(E))$ by

$$\Pi(\omega) = \begin{cases} \pi^{p,0}(\omega), & \text{if } p \le n, \text{ and} \\ I(\pi^{n,p-n}(\omega)), & \text{if } p > n. \end{cases} \tag{2.4}$$

If $\phi\colon E \to E$ is a projectable map, that is, if ϕ induces a map $\phi_0\colon M \to M$ such that

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ M & \xrightarrow{\phi_0} & M \end{array},$$

then the infinite prolongation

$$\operatorname{pr}\phi\colon J^\infty(E) \to J^\infty(E)$$

is defined by

$$\operatorname{pr}\phi(j^\infty(s)(x_0)) = j^\infty(\phi\circ s\circ\phi_0^{-1})(\phi_0(x_0)).$$

If X is a projectable vector field on E, then the flow ϕ_ϵ of X is a projectable map and the prolongation of X is the vector field $\operatorname{pr} X$ on $J^\infty(E)$ defined by

$$\operatorname{pr} X = \left.\frac{d\operatorname{pr}\phi_\epsilon}{d\epsilon}\right|_{\epsilon=0}.$$

Since $\operatorname{pr}\phi$ commutes with π_M^∞ and preserves the contact ideal on $J^\infty(E)$, the prolongation $\operatorname{pr}\phi$ preserves the decomposition (2.1), in the sense that

$$\operatorname{pr}\phi^*\colon \Omega^{r,s}(J^\infty(E)) \to \Omega^{r,s}(J^\infty(E)),$$

and, moreover, $\operatorname{pr}\phi$ commutes with the maps d_H, d_V and I. Consequently we may define

$$\Omega^{r,s}_{\operatorname{pr} G}(J^\infty(E)) = \{\omega\in\Omega^{r,s}_{\operatorname{pr} G}(J^\infty(E)) \mid \operatorname{pr}\phi^*(\omega)=\omega \text{ for all } \phi\in G\}, \tag{2.5}$$

the G invariant augmented variational bicomplex $\{\,\Omega^{*,*}_{\operatorname{pr} G}(J^\infty(E)), d_H, d_V\,\}$ and the G invariant Euler-Lagrange complex $\{\,\mathcal{E}^*_{\operatorname{pr} G}(J^\infty(E)), d_H, \delta_V\,\}$ on E by

$$\begin{tikzcd}
 & & {} & & & & {} & {} & \\
 & 0 \arrow[r] & \Omega^{0,3}_{\operatorname{pr} G} \arrow[u, "d_V"] & & \cdots & & \Omega^{n,3}_{\operatorname{pr} G} \arrow[u, "d_V"] \arrow[r, "I"] & \mathcal{F}^3_{\operatorname{pr} G} \arrow[u, "\delta_V"] \arrow[r] & 0 \\
 & 0 \arrow[r] & \Omega^{0,2}_{\operatorname{pr} G} \arrow[u, "d_V"] \arrow[r, "d_H"] & \Omega^{1,2}_{\operatorname{pr} G} \arrow[r, "d_H"] & \cdots & \Omega^{n-1,2}_{\operatorname{pr} G} \arrow[r, "d_H"] & \Omega^{n,2}_{\operatorname{pr} G} \arrow[u, "d_V"] \arrow[r, "I"] & \mathcal{F}^2_{\operatorname{pr} G} \arrow[u, "\delta_V"] \arrow[r] & 0 \\
 & 0 \arrow[r] & \Omega^{0,1}_{\operatorname{pr} G} \arrow[u, "d_V"] \arrow[r, "d_H"] & \Omega^{1,1}_{\operatorname{pr} G} \arrow[u, "d_V"] \arrow[r, "d_H"] & \cdots & \Omega^{n-1,1}_{\operatorname{pr} G} \arrow[u, "d_V"] \arrow[r, "d_H"] & \Omega^{n,1}_{\operatorname{pr} G} \arrow[u, "d_V"] \arrow[r, "I"] & \mathcal{F}^1_{\operatorname{pr} G} \arrow[u, "\delta_V"] \arrow[r] & 0 \\
0 \arrow[r] & \mathbf{R} \arrow[r] & \Omega^{0,0}_{\operatorname{pr} G} \arrow[u, "d_V"] \arrow[r, "d_H"] & \Omega^{1,0}_{\operatorname{pr} G} \arrow[u, "d_V"] \arrow[r, "d_H"] & \cdots & \Omega^{n-1,0}_{\operatorname{pr} G} \arrow[u, "d_V"] \arrow[r, "d_H"] & \Omega^{n,0}_{\operatorname{pr} G} \arrow[u, "d_V"] & &
\end{tikzcd}$$

and

$$0 \longrightarrow \mathbf{R} \longrightarrow \Omega^{0,0}_{\operatorname{pr} G} \xrightarrow{d_H} \Omega^{1,0}_{\operatorname{pr} G} \xrightarrow{d_H} \cdots$$
$$\xrightarrow{d_H} \Omega^{n-1,0}_{\operatorname{pr} G} \xrightarrow{d_H} \Omega^{n,0}_{\operatorname{pr} G} \xrightarrow{E} \mathcal{F}^1_{\operatorname{pr} G} \xrightarrow{\delta_V} \mathcal{F}^2_{\operatorname{pr} G} \xrightarrow{\delta_V} \mathcal{F}^3_{\operatorname{pr} G} \xrightarrow{\delta_V} \cdots .$$

It is our contention, which is supported by a wide variety of examples and applications, that G invariant variational bicomplexes and G invariant Euler-Lagrange complexes play a central role in many areas of differential geometry and mathematical physics and in the applications of Lie groups to differential equations.

To describe the variational bicomplex in coordinates, let us introduce local coordinates $(x^i, u^\alpha) \to (x^i)$, $i = 1, 2, \dots, n$ and $\alpha = 1, 2, \dots, m$ for the fiber bundle E. The adapted coordinates for $J^\infty(E)$ are then

$$(x^i, u^\alpha, u^\alpha_{i_1}, u^\alpha_{i_1 i_2}, \dots, u^\alpha_{i_1 i_2 \cdots i_k}, \dots) \tag{2.6}$$

where, as usual, if $s^\alpha = s^\alpha(x^i)$ is a local section of E and $\sigma = j^\infty(s)(x_0)$, then

$$u^\alpha_{i_1 i_2 \cdots i_k}(\sigma) = \frac{\partial^k s^\alpha}{\partial x^{i_1} \partial x^{i_2} \cdots \partial x^{i_k}}(x_0).$$

In these coordinates the contact ideal on $J^\infty(E)$ is generated by the contact 1 forms

$$\theta^\alpha_{i_1 i_2 \cdots i_k} = du^\alpha_{i_1 i_2 \cdots i_k} - u^\alpha_{i_1 i_2 \cdots i_k j}\, dx^j.$$

A vector X on $J^\infty(E)$ is a total vector if it annihilates all contact 1 forms — locally all total vector fields on $J^\infty(E)$ assume the form

$$X = A^i\, D_i,$$

where the coefficients A^i are smooth functions of finitely many of the coordinates (2.6) and D_i is the total derivative operator with respect to x^i, namely

$$D_i = \frac{\partial}{\partial x^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} + u^\alpha_{ij} \frac{\partial}{\partial u^\alpha_j} + u^\alpha_{ijk} \frac{\partial}{\partial u^\alpha_{jk}} + \cdots .$$

A form $\omega \in \Omega^{r,s}(J^\infty(E))$ is locally a finite sum of terms

$$f\, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_r} \wedge \theta^{\alpha_1}_{j_{11} j_{12} \cdots j_{1k_1}} \wedge \theta^{\alpha_2}_{j_{21} j_{22} \cdots j_{2k_2}} \wedge \cdots \wedge \theta^{\alpha_s}_{j_{s1} j_{s2} \cdots j_{sk_s}},$$

where f is a smooth function of finitely many of the variables (2.6). The d_H and d_V structure equations are

$$d_H(f) = D_i(f)\, dx^i, \qquad d_V(f) = \sum_{k=0}^\infty \frac{\partial f}{\partial u^\alpha_{i_1 i_2 \cdots i_k}}\, \theta^\alpha_{i_1 i_2 \cdots i_k},$$

$$d_H(dx^i) = 0, \qquad d_V(dx^i) = 0,$$

$$d_H(\theta^\alpha_{i_1 i_2 \cdots i_k}) = dx^l \wedge \theta^\alpha_{l i_1 i_2 \cdots i_k}, \qquad d_V(\theta^\alpha_{i_1 i_2 \cdots i_k}) = 0.$$

Finally, the integration by parts operator $I \colon \Omega^{n,s}(J^\infty(E)) \to \Omega^{n,s}(J^\infty(E))$ is defined by

$$I(\omega) = \frac{1}{s}\, \theta^\alpha \wedge \big[\big(\frac{\partial}{\partial u^\alpha} \lrcorner\, \omega \big) - D_i \big(\frac{\partial}{\partial u^\alpha_i} \lrcorner\, \omega \big) + D_{ij} \big(\frac{\partial}{\partial u^\alpha_{ij}} \lrcorner\, \omega \big) - \cdots \big].$$

It is apparent from this formula that $\delta_V : \Omega^{n,0}(J^\infty(E)) \to \mathcal{F}^1(J^\infty(E))$ gives precisely the classical Euler-Lagrange operator from the calculus of variations.

§3. Cohomology of Lie algebras of formal infinitesimal vector fields. Let G be a Lie pseudo-group acting on $\mathbf{R}^N$ and let Γ be the Lie algebra of infinitesimal generators of G. From the definition of a Lie pseudo-group, it follows that the vector fields $X \in \Gamma$ are characterized as the solutions of some system of linear, homogeneous, involutive, partial differential equations. More precisely, let $J^k(T\mathbf{R}^N)$ be the k^{th} order jet bundle of the tangent bundle $T\mathbf{R}^N \to \mathbf{R}^N$ and let $\mathcal{R}^k \hookrightarrow J^k(T\mathbf{R}^N)$ be the linear subspace of $J^k(T\mathbf{R}^N)$ defined by these linear partial differential equations. The assumption of involutivity implies that, for $l \geq k$, the projection maps

$$\pi_l^{l+1} : J^{l+1}(T\mathbf{R}^N) \to J^l(T\mathbf{R}^N)$$

restrict to surjective maps

$$\pi_l^{l+1} : \mathcal{R}^{l+1} \to \mathcal{R}^l,$$

where $\mathcal{R}^l$ is the $(l-k)^{\text{th}}$ prolongation of $\mathcal{R}^k$. Let $\mathcal{R}^\infty \hookrightarrow J^\infty(E)$ be the infinite prolongation of $\mathcal{R}^k$. A vector field X on $\mathbf{R}^N$ then belongs to Γ if and only if, for each point $p \in \mathbf{R}^N$, $j^\infty(X)(p) \in \mathcal{R}^\infty$.

Next we let $\mathcal{X}_{\text{frm}}(\mathbf{R}^N)$ be the topological Lie algebra of formal vector fields on $\mathbf{R}^N$ — if $X \in \mathcal{X}_{\text{frm}}(\mathbf{R}^N)$, then we can write

$$X = a^i(x^j)\frac{\partial}{\partial x^i}, \tag{3.1}$$

where the coefficients $a^i(x^j)$ are given by formal power series

$$a^i(x^j) = \sum_{k=0}^{\infty} a^i_{j_1 j_2 \cdots j_k} x^{j_1} x^{j_2} \cdots x^{j_k}. \tag{3.2}$$

The series (3.1) need not have a positive radius of convergence. Nevertheless there still exists a smooth vector field $\widetilde{X} = \widetilde{X}^i \dfrac{\partial}{\partial x^i}$ on $\mathbf{R}^N$ such that

$$a^i_{j_1 j_2 \cdots j_k} = \frac{\partial^k \widetilde{X}^i}{\partial x^{j_1} \partial x^{j_2} \cdots \partial x^{j_k}}(0)$$

and hence there is a natural identification

$$\mathcal{X}_{\text{frm}}(\mathbf{R}^N) \cong J^\infty(T\mathbf{R}^N)(0).$$

We define the formal infinitesimal Lie pseudo-group Γ_{frm} associated to the Lie pseudo-group G by

$$\Gamma_{\text{frm}} = \{\, X \in \mathcal{X}_{\text{frm}}(\mathbf{R}^N) \mid X = j^\infty(\widetilde{X})(0) \text{ and } j^\infty(\widetilde{X})(0) \in \mathcal{R}^\infty \,\}.$$

As a simple example, the formal infinitesimal Lie pseudo-group associated to the Lie pseudo-group of local diffeomorphisms which preserve the volume form

$$\nu = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^N$$

is the infinite dimensional Lie algebra of formal vector fields (3.1) whose coefficients (3.2) satisfy the trace condition

$$\sum_{i=1}^{N} a^i_{ij_2\cdots j_k} = 0.$$

The exterior algebra $A^*(\mathcal{X}_{\mathrm{frm}}(\mathbf{R}^N))$ of continuous cochains on $\mathcal{X}_{\mathrm{frm}}(\mathbf{R}^N)$ is the real exterior algebra generated by the 1 cochains

$$\delta^i_{j_1j_2\cdots j_k}(X) = (-1)^k a^i_{j_1j_2\cdots j_k} = \frac{(-1)^k}{k!}\frac{\partial^k}{\partial x^{j_1}\partial x^{j_2}\cdots\partial x^{j_k}}\langle X, dx^i\rangle.$$

The exterior derivative

$$d\colon A^p(\mathcal{X}_{\mathrm{frm}}(\mathbf{R}^N)) \to A^{p+1}(\mathcal{X}_{\mathrm{frm}}(\mathbf{R}^N))$$

is the usual Lie algebra cohomology differential as defined in terms of the Lie bracket of formal vector fields. The differential d is an anti-derivation and is defined on 1 chains $\omega \in A^1(\mathcal{X}_{\mathrm{frm}}(\mathbf{R}^N))$ by

$$d\,\omega(X,Y) = -\omega([X,Y]).$$

If $Z \in \mathcal{X}_{\mathrm{frm}}(\mathbf{R}^N)$ and $\omega \in A^p(\mathcal{X}_{\mathrm{frm}}(\mathbf{R}^N))$, then the Lie derivative of ω with respect to Z is

$$(\mathcal{L}_Z\omega)(X_1, X_2, \ldots, X_p) = \sum_{i=1}^{p}(-1)^i\omega([Z, X_i], X_1, \ldots, \widehat{X_i}, \ldots, X_p).$$

We define $A^*(\Gamma_{\mathrm{frm}})$ to be the restriction of $A^*(\mathcal{X}_{\mathrm{frm}}(\mathbf{R}^N))$ to the subalgebra $\Gamma_{\mathrm{frm}} \subset \mathcal{X}_{\mathrm{frm}}$ — if $\omega \in A^*(\Gamma_{\mathrm{frm}})$ then there is a cochain $\widetilde{\omega} \in A^*(\mathcal{X}_{\mathrm{frm}})$ such that

$$\omega(X_1, X_2, \ldots, X_p) = \widetilde{\omega}(X_1, X_2, \ldots, X_p),$$

for all X_1, X_2, ..., $X_p \in \Gamma_{\mathrm{frm}}$.

Let Γ^0_{frm} be a Lie subalgebra of Γ_{frm}. The Lie algebra Γ^0_{frm} need not be a formal infinitesimal Lie pseudo-group in its own right. Then the Γ^0_{frm} relative cochains on Γ_{frm} are defined by

$$A^p(\Gamma_{\mathrm{frm}}, \Gamma^0_{\mathrm{frm}}) = \{\, \omega \in A^p(\Gamma_{\mathrm{frm}}) \mid X \lrcorner\, \omega = 0 \text{ and } \mathcal{L}_X\omega = 0 \text{ for all } X \in \Gamma^0_{\mathrm{frm}} \,\}.$$

The Lie algebra differential d restricts to the subcomplex $A^*(\Gamma_{\mathrm{frm}}, \Gamma^0_{\mathrm{frm}})$ and we call the vector spaces

$$H^p(A^*(\Gamma_{\mathrm{frm}}, \Gamma^0_{\mathrm{frm}}))$$

the p^{th} Lie algebra cohomology of the Lie algebra of formal vector fields Γ_{frm} relative to Γ^0_{frm}.

We apply these considerations to the fiber bundle $E\colon \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ (with coordinates $(x^i, u^\alpha) \to (x^i)$) and the pseudo-group of projectable transformations on E. A formal projectable vector field $X \in \Gamma_{\text{frm,prj}}(E)$ takes the form

$$X = a^i(x^j)\frac{\partial}{\partial x^i} + b^\alpha(x^j, u^\beta)\frac{\partial}{\partial u^\alpha},$$

where the coefficients $a^i(x^j)$ and $b^\alpha(x^j, u^\beta)$ are given by formal power series

$$a^i(x^j) = \sum_{k=0}^{\infty} a^i_{j_1 j_2 \cdots j_k} x^{j_1} x^{j_2} \cdots x^{j_k}$$

and

$$b^\alpha(x^j, u^\beta) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} b^\alpha_{j_1 j_2 \cdots j_k \beta_1 \beta_2 \cdots \beta_l} x^{j_1} x^{j_2} \cdots x^{j_k} u^{\beta_1} u^{\beta_2} \cdots u^{\beta_l}.$$

The generators for the exterior algebra $A^*(\mathcal{X}_{\text{frm,prj}}(E))$ are then

$$\delta^i_{j_1 j_2 \cdots j_k}(X) = (-1)^k a^i_{j_1 j_2 \cdots j_k} \tag{3.3}$$

and

$$\delta^\alpha_{j_1 j_2 \cdots j_k \beta_1 \beta_2 \cdots \beta_l}(X) = (-1)^{k+l} b^\alpha_{j_1 j_2 \cdots j_k \beta_1 \beta_2 \cdots \beta_l}. \tag{3.4}$$

If G is a Lie pseudo-group of projectable transformations on E with associated formal infinitesimal Lie pseudo-group $\mathcal{X}_{\text{frm}}$, then we define the linear isotropy subalgebra of Γ_{frm} at a point $\sigma \in J^\infty(E)$ by

$$\Gamma^0_{\text{frm}}(\sigma) = \{\, X \in \Gamma_{\text{frm}} \mid \operatorname{pr} X(\sigma) = 0 \,\}. \tag{3.5}$$

The bracket formula

$$\operatorname{pr}[\,X, Y\,] = [\,\operatorname{pr} X, \operatorname{pr} Y\,]$$

insures that $\Gamma^0_{\text{frm}}(\sigma)$ is a subalgebra of Γ_{frm}.

§4. The evaluation map. For the moment, let G be a finite dimensional Lie transformation group acting on a manifold M and let $\Omega^p_G(M)$ be the space of G invariant p forms on M. Let Γ be the Lie algebra of infinitesimal generators for the action of G on M. We define a map

$$\varrho\colon \Omega^p_G(M) \times M \to A^p(\Gamma)$$

by setting, for $\omega \in \Omega^p_G(M)$, $m \in M$ and $X_1, X_2, \ldots, X_p \in \Gamma$,

$$\varrho(\omega, m)(X_1, X_2, \ldots, X_p) = (-1)^p \omega(X_1, X_2, \ldots, X_p)(m). \tag{4.1}$$

It is not difficult to show that ϱ enjoys the following properties.

(i) For fixed m, the map ϱ defines a cochain map between the G invariant de Rham complex on $\Omega^*_G(M)$ and the complex of real cochains on Γ, that is, for $\omega \in \Omega^p_G(M)$,

$$\varrho(d\omega, m) = d(\varrho(\omega, m)). \tag{4.2}$$

(ii) The map ϱ transforms equivariantly under the adjoint action of G on Γ so that if $g \in G$, then

$$\varrho(\omega, g \cdot m) = \mathrm{Ad}^*_g \varrho(\omega, m). \tag{4.3}$$

(iii) If we define the linear isotropy subalgebra $\Gamma^0(m)$ at the point $m \in \Gamma$ by

$$\Gamma^0(m) = \{\, X \in \Gamma \mid X(m) = 0 \,\}$$

then, for each $\omega \in \Omega^p_G(M)$ and $m \in M$, it follows that

$$\varrho(\omega, m) \in A^p(\Gamma, \Gamma^0(m)).$$

An important example of these general considerations arise when (M, ω) is a symplectic manifold and G is a Lie group of symplectomorphisms ([8]). Then $\varrho_\omega(m) = \varrho(\omega, m)$ defines a map

$$\varrho_\omega : M \to Z^2(\mathbf{g})$$

from M to the closed 2 cochains on the Lie algebra $\mathbf{g}$ of G.

Now consider the situation where G is an infinite dimensional Lie pseudo-group of projectable transformations acting on the fiber bundle $\pi \colon E \to M$. Then we can apply these same considerations to define a map

$$\varrho \colon \Omega^p_{\mathrm{pr}\,G}(J^\infty(E)) \times J^\infty(E) \to A^p(\Gamma_{\mathrm{frm}})$$

provided we first prolong the vector fields in (4.1) so that, for $\omega \in \Omega^p_{\mathrm{pr}\,G}(J^\infty(E))$, $\sigma \in J^\infty(E)$, and $X_1, X_2, \ldots, X_p \in \Gamma_{\mathrm{frm}}$,

$$\varrho(\omega, \sigma)(X_1, X_2, \ldots, X_p) = (-1)^p \omega(\mathrm{pr}\, X_1, \mathrm{pr}\, X_2, \ldots, \mathrm{pr}\, X_p)(\sigma). \tag{4.4}$$

We can prove, for a wide variety of Lie pseudo-groups G, *that for generically chosen points $\sigma \in J^\infty(E)$ there then exists an open neighborhood $\mathcal{U}$ of σ in $J^\infty(E)$ and an isomorphism*

$$\Phi \colon H^p(\mathcal{E}^*_{\mathrm{pr}\,G}(\mathcal{U})) \to H^p(A^*(\Gamma_{\mathrm{frm}}, \Gamma^0_{\mathrm{frm}}(\sigma))) \tag{4.5}$$

in cohomology. The construction of this isomorphism involves two principal steps.

Step A. We first prove that the interior rows of the $\operatorname{pr} G$ invariant variational bicomplex

$$0 \longrightarrow \Omega^{0,s}_{\operatorname{pr} G}(\mathcal{U}) \xrightarrow{d_H} \Omega^{1,s}_{\operatorname{pr} G}(\mathcal{U}) \xrightarrow{d_H} \cdots \Omega^{n-1,s}_{\operatorname{pr} G}(\mathcal{U}) \xrightarrow{d_H} \Omega^{n,s}_{\operatorname{pr} G}(\mathcal{U}) \xrightarrow{I} \mathcal{F}^{s}_{\operatorname{pr} G}(\mathcal{U}) \to 0 \tag{4.6}$$

are exact. For the invariant variational bicomplexes we study in sections 5 and 6 this step is quite easy. In general we prove (4.6) is exact by constructing an $\operatorname{pr} G$ invariant connection on the vector bundle of total vector fields on $J^\infty(E)$. It follows from the exactness of (4.6) that the projection map (2.4) defines an isomorphism

$$\Pi\colon H^p(\mathcal{E}^*_{\operatorname{pr} G}(\mathcal{U})) \to H^p(\Omega^*_{\operatorname{pr} G}(\mathcal{U})).$$

The isomorphism Φ is then defined by

$$\Phi(\omega) = \varrho_\sigma \circ \Pi^{-1}(\omega). \tag{4.7}$$

Step B. In this step we construct a subbundle $\mathcal{K}^\infty \subset \mathcal{U}$ with the properties that

(i) $\operatorname{pr} G$ acts transitively on $\mathcal{K}^\infty$; and

(ii) $\mathcal{K}^\infty$ is $\operatorname{pr} G$ invariant strong deformation retract of $\mathcal{U}$, that is, there is a smooth map

$$H : \mathcal{U} \times [0,1] \to \mathcal{U}$$

such that

$$\begin{aligned} H(\sigma, 0) &= \sigma && \text{for all } \sigma \in \mathcal{U}, \\ H(\sigma, 1) &\in \mathcal{K}^\infty && \text{for all } \sigma \in \mathcal{U}, \\ H(\sigma, t) &= \sigma && \text{for all } \sigma \in \mathcal{K}^\infty \text{ and } t \in [0,1], \text{ and} \\ H(\operatorname{pr} g \cdot \sigma, t) &= \operatorname{pr} g \cdot H(\sigma, t) && \text{for all } \sigma \in \mathcal{U},\ t \in [0,1] \text{ and } g \in G. \end{aligned}$$

Under these conditions it is not difficult to prove that the inclusion map

$$\iota : \mathcal{K}^\infty \to \mathcal{U}$$

and the evaluation map

$$\varrho\colon \Omega^p_{\operatorname{pr} G}(\mathcal{K}^\infty) \to A^p(\Gamma_{\mathrm{frm}}, \Gamma^0_{\mathrm{frm}}(\sigma))$$

induce isomorphisms in cohomology which, together with (4.6), suffice to establish the isomorphism (4.7). In the next two sections we carry out these two steps for two different invariant Euler-Lagrange complexes and we explicitly compute the isomorphism (4.7).

For finite dimensional Lie groups G, the isomorphism (4.7) was established in [3].

§5. The natural Euler-Lagrange complex for regular plane and space curves. We consider first the case of regular plane curves and accordingly we take E to be the trivial bundle $E\colon \mathbf{R}\times\mathbf{R}^2 \to \mathbf{R}$. We shall refer to E with local coordinates (t, R), where $R = x\,\mathbf{i} + y\,\mathbf{j} \in \mathbf{R}^2$. Let G be the Lie pseudo-group $G = \mathrm{Diff}(\mathbf{R})^+ \times \mathrm{Eucl}(2)$ acting on E by

$$(\phi, A, b)\cdot(t, R) = (\phi(t), AR + b), \tag{5.1}$$

where ϕ is a smooth function on $\mathbf{R}$, $\phi' > 0$, $A \in \mathrm{SO}(2)$, and $b \in \mathbf{R}^2$. The Lie algebra Γ of infinitesimal generators of G is spanned by the vector fields

$$X_f = f(t)\frac{\partial}{\partial t}, \qquad T_x = \frac{\partial}{\partial x}, \qquad T_y = \frac{\partial}{\partial y}, \qquad R_{xy} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \tag{5.2}$$

We define an open set in $J^\infty(E)$ by

$$\mathcal{P}^\infty = \{\, (t, R, \dot R, \ddot R, \dddot R, \dots) \in J^\infty(E) \mid \dot R \neq 0 \,\}.$$

It is easy to see that the prolonged action of G to $J^\infty(E)$ preserves $\mathcal{P}^\infty$. The Euler-Lagrange complex

$$\Omega^{0,0}_{\mathrm{pr}\,G}(\mathcal{P}^\infty) \xrightarrow{d_H} \Omega^{1,0}_{\mathrm{pr}\,G}(\mathcal{P}^\infty) \xrightarrow{\delta_V} \mathcal{F}^1_{\mathrm{pr}\,G}(\mathcal{P}^\infty) \xrightarrow{\delta_V} \mathcal{F}^2_{\mathrm{pr}\,G}(\mathcal{P}^\infty) \xrightarrow{\delta_V} \cdots$$

is called the *natural Euler-Lagrange complex* $\mathcal{E}^*_{\mathrm{pr}\,G}(\mathcal{P}^\infty)$ *for regular plane curves.* We remark that if $\lambda \in \Omega^{1,0}_{\mathrm{pr}\,G}(\mathcal{P}^\infty)$ is any Lagrangian, then its value $(j^\infty(c)(t))^*(\lambda)$ on a regular curve $t \to c(t)$ is independent of the curve's parametrization, position and orientation in the plane. Moreover if λ defines a non-trivial class in $H^1(\mathcal{E}^*_{\mathrm{pr}\,G}(\mathcal{P}^\infty))$ and $c_1 \colon S^1 \to \mathbf{R}^2$ and $c_2 \colon S^1 \to \mathbf{R}^2$ are two regular isotopic plane curves then the first variational formula implies that the one forms $(j^\infty(c_1)(t))^*(\lambda)$ and $(j^\infty(c_2)(t))^*(\lambda)$ define the same class in $H^1_{\text{de Rham}}(S^1)$.

The interior rows of the natural variational bicomplex for plane curves are just the short sequences

$$0 \to \Omega^{0,s}_{\mathrm{pr}\,G}(\mathcal{P}^\infty) \xrightarrow{d_H} \Omega^{1,s}_{\mathrm{pr}\,G}(\mathcal{P}^\infty) \xrightarrow{I} \mathcal{F}^s_{\mathrm{pr}\,G}(\mathcal{P}^\infty) \to 0, \tag{5.3}$$

where $s \geq 1$. The exactness of (5.3) follows immediately from the exactness of the interior rows of the *free* variational bicomplex for regular plane curves

$$0 \to \Omega^{0,s}(\mathcal{P}^\infty) \xrightarrow{d_H} \Omega^{1,s}(\mathcal{P}^\infty) \xrightarrow{I} \mathcal{F}^s(\mathcal{P}^\infty) \to 0. \tag{5.4}$$

Indeed, if $\omega \in \Omega^{1,s}_{\mathrm{pr}\,G}(\mathcal{P}^\infty)$ satisfies $I(\omega) = 0$, then there is a form $\eta \in \Omega^{0,s}(\mathcal{P}^\infty)$ such that $\omega = d_H\eta$. That η is necessarily $\mathrm{pr}\,G$ invariant follows from the exactness of (5.4) at $\Omega^{0,s}(\mathcal{P}^\infty)$ — if $\psi \in G$, then

$$d_H[\mathrm{pr}\,\psi^*(\eta) - \eta] = \mathrm{pr}\,\psi^*(d_H\eta) - d_H(\eta) = \mathrm{pr}\,\psi^*(\omega) - \omega = 0,$$

and hence $\operatorname{pr}\psi^*(\eta) = \eta$. This completes the step A in the proof of the isomorphism (4.7).

The classical local differential geometry of regular plane curves suggests the way to construct the subbundle $\mathcal{K}^\infty \subset \mathcal{P}^\infty$ and the $\operatorname{pr} G$ invariant deformation retraction H. We introduce the following adapted coordinates for $\mathcal{P}^k$:

$$\begin{aligned} \mathcal{P}^1 &= \{\, (t, R, (T, N), v) \,\}, \\ \mathcal{P}^2 &= \{\, (t, R, (T, N), v, v', \kappa) \,\}, \\ \mathcal{P}^3 &= \{\, (t, R, (T, N), v, v', \kappa, v'', \kappa') \,\}, \quad \dots, \\ \mathcal{P}^{k+1} &= \{\, (t, R, (T, N), v, v', \kappa, v'', \kappa', \dots, v^{(k)}, \kappa^{(k-1)}, \dots) \,\}, \end{aligned} \tag{5.5a}$$

where

$$\begin{gathered} v = \sqrt{\dot{x}^2 + \dot{y}^2}, \qquad T = \frac{\dot{x}\,\mathbf{i} + \dot{y}\,\mathbf{j}}{v}, \qquad N = \frac{-\dot{y}\,\mathbf{i} + \dot{x}\,\mathbf{j}}{v}, \\ \kappa = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{v^3}, \qquad \text{and} \qquad ' = \frac{d}{ds} = \frac{1}{v}\frac{d}{dt}. \end{gathered} \tag{5.5b}$$

We have enclosed the Frenet frame (T, N) in parenthesis to indicate the fact that this frame defines just a single jet coordinate. Note that $v > 0$. We remark that $\dfrac{d}{ds}$ is an invariant total differential operator in that

$$[\, \frac{d}{ds}, \operatorname{pr} X \,] = 0, \qquad \text{for all } X \in \Gamma.$$

The functions κ, κ', κ'', ..., are all $\operatorname{pr} G$ invariant functions and so the action of $\operatorname{pr} G$ on $\mathcal{P}^\infty$ is given in terms of the coordinates (5.5) by

$$\begin{aligned} &(\phi, A, b)\cdot(t, R, (T, N), v, v', \kappa, v'', \kappa', \dots, v^{(k)}, \kappa^{(k-1)}, \dots) \\ &\qquad (\phi(t), AR + b, (AT, AN), \tilde{v}, \tilde{v}', \kappa, \tilde{v}'', \kappa', \dots, \tilde{v}^{(k)}, \kappa^{(k-1)}, \dots), \end{aligned} \tag{5.6}$$

where

$$\tilde{v} = \frac{1}{\dot{\phi}} v = \frac{1}{\phi'}, \quad \text{and} \quad \tilde{v}^k = \frac{d^k}{ds^k}\Big(\frac{1}{\phi'}\Big). \tag{5.7}$$

We let

$$\mathcal{K}^\infty = \{\, \sigma \in \mathcal{P}^\infty \mid \kappa(\sigma) = 0,\ \kappa'(\sigma) = 0,\ \kappa''(\sigma) = 0, \dots \,\}.$$

It is clear from (5.6) and (5.7) that $\operatorname{pr} G$ acts transitively on $\mathcal{F}^\infty$ and that the map

$$H : \mathcal{P}^\infty \times [0, 1] \to \mathcal{P}^\infty$$

defined by

$$H(\sigma, \epsilon) = (t, R, (T, N), v, v', \epsilon\kappa, v'', \epsilon\kappa', \dots, v^{(k)}, \epsilon\kappa^{(k-1)}, \dots)$$

is a $\operatorname{pr} G$ equivariant strong deformation retraction of $\mathcal{P}^\infty$ onto $\mathcal{F}^\infty$. This completes Step B described in the general considerations of the previous section and thus

$$H^*(\mathcal{E}^*_{\operatorname{pr} G}(\mathcal{P}^\infty)) \cong H^*(A^*(\Gamma_{\mathrm{frm}}, \Gamma^0_{\mathrm{frm}}(\sigma_0))). \tag{5.8}$$

To complete this example, we need to compute the Lie algebra cohomology of the complex $A^*(\Gamma_{\mathrm{frm}}, \Gamma^0_{\mathrm{frm}}(\sigma_0))$ and then to construct the pull-backs of the representatives of the Lie algebra cohomology classes in $A^*(\Gamma_{\mathrm{frm}}, \Gamma^0_{\mathrm{frm}}(\sigma_0))$ to forms in $\mathcal{E}^*_{\operatorname{pr} G}(\mathcal{P}^\infty)$. To begin, we note that the 1 cochains in $A^*(\mathcal{X}_{\mathrm{frm}}(E))$ are

$$\delta^t_{t^k x^l y^m}, \quad \delta^x_{t^k x^l y^m}, \quad \text{and} \quad \delta^y_{t^k x^l y^m},$$

but that among these only the cochains

$$\delta^t_{t^k}, \quad \delta^x, \quad \delta^y, \quad \text{and} \quad \vartheta_{xy} = \frac{1}{2}(-\delta^x_y + \delta^y_x)$$

are independent in Γ_{frm}. These cochains generate $A^*(\Gamma_{\mathrm{frm}})$.

Next we compute $\Gamma^0_{\mathrm{frm}}(\sigma_0)$, where the point $\sigma_0 \in \mathcal{K}^\infty$ has coordinates

$$t = 0, \quad R = 0, \quad (T, N) = (\mathbf{i}, \mathbf{j}), \quad v = 1, \quad v^{(k)} = 0, \quad k \geq 1.$$

Given the formal vector field

$$X = X_f + a\, T_x + b\, T_y + c\, R_{xy},$$

we compute

$$\operatorname{pr} X(t)(\sigma_0) = f(0), \quad \operatorname{pr} X(R)(\sigma_0) = a\,\mathbf{i} + b\,\mathbf{j}, \quad \operatorname{pr} X(T,N)(\sigma_0) = c\,(\mathbf{j}, \mathbf{i}),$$
$$\operatorname{pr} X(v)(\sigma_0) = f'(0), \quad \text{and} \quad \operatorname{pr} X(v^{(k)})(\sigma_0) = f^{(k+1)}(0).$$

In evaluating $\operatorname{pr} X(v^{(k)})(\sigma_0)$ we make use of the fact that the $\operatorname{pr} X$ commutes with $\dfrac{d}{ds}$. It now follows that if $\operatorname{pr} X(\sigma_0) = 0$, then $a = b = c = 0$ and $f^k(0) = 0$ for all $k = 0, 1, 2, \ldots$. The formal vector field X therefore vanishes and the linear infinitesimal isotropy subalgebra $\Gamma^0_{\mathrm{frm}}(\sigma_0) = \{\, 0 \,\}$.

Since $\Gamma_{\mathrm{frm}} = \mathcal{X}_{\mathrm{frm}}(\mathbf{R}) \times \mathrm{eucl}(2)$, where $\mathrm{eucl}(2)$ is the Lie algebra of the Euclidean group in the plane, we deduce that

$$H^*(A^*(\Gamma_{\mathrm{frm}})) = H^*(\mathcal{X}_{\mathrm{frm}}(\mathbf{R})) \otimes H^*(\mathrm{eucl}(2)). \tag{5.9}$$

The Lie algebra cohomology of $\mathrm{eucl}(2)$ is easily computed directly to have representatives

$$\vartheta_{xy}, \quad \delta^x \wedge \delta^y, \quad \delta^x \wedge \delta^y \wedge \vartheta_{xy}. \tag{5.10}$$

The cohomology of the Lie algebra $\mathcal{X}_{\rm frm}(\mathbf{R})$ of formal vector fields on the line is also well known and can be computed by first noting that the radial vector field $Z = t\dfrac{d}{dt}$ acts on the generators $\delta^t_{t^n}$ for $A^*(\mathcal{X}_{\rm frm}(R))$ according to

$$\mathcal{L}_Z \delta^t_{t^k} = (1-k)\delta^t_{t^k}.$$

The spaces $A^p(\mathcal{X}_{\rm frm}(\mathbf{R}))$ therefore decompose into a direct sum of the eigenspaces $A^p_w(\mathcal{X}_{\rm frm}(\mathbf{R}))$ of $\mathcal{L}_Z$ with eigenvalue w. Each of these eigenspaces is preserved by the exterior derivative in the sense that

$$d\colon A^p_w(\mathcal{X}_{\rm frm}(\mathbf{R})) \to A^{p+1}_w(\mathcal{X}_{\rm frm}(\mathbf{R}))$$

and, moreover, the Lie derivative formula

$$\mathcal{L}_Z(\omega) = d(Z \lrcorner\, \omega) + Z \lrcorner\, d\omega$$

proves that every cohomology class in $A^p(\mathcal{X}_{\rm frm}(\mathbf{R}))$ has a representative which is an eigenform of $\mathcal{L}_Z$ with eigenvalue 0. From this observation it is a simple matter to check that the Lie algebra cohomology of $\mathcal{X}_{\rm frm}(\mathbf{R})$ is generated by the single 3 cochain

$$\gamma_{\rm GF} = \delta^t \wedge \delta^t_t \wedge \delta^t_{t^2}. \tag{5.11}$$

We hasten to remark that the computation of $H^*(A^*(\mathcal{X}_{\rm frm}(\mathbf{R}^n)))$ is, for $n \geq 1$, considerably less trivial ([5], [7]).

From equations (5.9), (5.10), and (5.11) we can now conclude that representatives for $H^p(A^*(\mathcal{X}_{\rm frm}))$ are

$$\begin{gathered} \vartheta_{xy}, \quad \delta^x \wedge \delta^y, \quad \delta^x \wedge \delta^y \wedge \vartheta_{xy}, \quad \gamma_{\rm GF}, \\ \vartheta_{xy} \wedge \gamma_{\rm GF}, \quad \delta^x \wedge \delta^y \wedge \gamma_{\rm GF}, \quad \delta^x \wedge \delta^y \wedge \vartheta_{xy} \wedge \gamma_{\rm GF}. \end{gathered} \tag{5.12}$$

Even though these Lie algebra cohomology calculations are quite elementary and based upon the simplest of techniques, the important point for us here is that the applicability of such methods is certainly not apparent, either in the original setting of the Euler-Lagrange complex or even in the exterior differential systems approach to this problem [6].

Finally, to pull-back the cochains (5.12) to the Euler-Lagrange complex we first need to find d closed forms in $\Omega^*_{\operatorname{pr} G}(\mathcal{P}^\infty)$ which map under the evaluation map ϱ to these cochains. We use the coordinate system (5.5) to define invariant forms

$$\tau = v\, dt, \quad \alpha = \frac{1}{v}\, d_V v, \quad \alpha^k = \frac{d^k}{ds^k}\alpha, \quad \gamma^k = d\kappa^{(k)},$$

and

$$\beta^1 = T \cdot dR, \quad \beta^2 = N \cdot dR, \quad \beta^3 = N \cdot dT$$

on $\mathcal{P}^\infty$. In the definition of α, it is necessary to use the vertical differential d_V in order that α be $\operatorname{pr} G$ invariant. It is easy to see that

$$\varrho(\tau) = \delta_0^t, \quad \varrho(\alpha^n) = (-1)^{n+1}\delta_{t^n}^t, \quad \text{and}$$

$$\varrho(\beta^1) = \delta^x, \quad \varrho(\beta^2) = \delta^y, \quad \varrho(\beta^3) = \vartheta_{xy}$$

and that

$$d\,\tau = \alpha \wedge \tau, \qquad d\,\alpha = \tau \wedge \alpha^1, \qquad d\,\alpha^1 = \tau \wedge \alpha^2,$$

$$d\,\beta^1 = \beta^2 \wedge \beta^3, \quad d\,\beta^2 = \beta^1 \wedge \beta^3, \quad d\,\beta^3 = 0.$$

It now follows that the isomorphism (5.8) is given explicitly by

$$\begin{aligned}
\vartheta_{xy} &\longleftrightarrow \lambda = \pi^{1,0}(\beta^3) = \kappa\,\tau = \frac{\dot x \ddot y - \ddot x \dot y}{v^2}\,dt,\\
\delta_0^x \wedge \delta_0^y &\longleftrightarrow \delta = I \circ \pi^{1,1}(\beta^1 \wedge \beta^2) = \tau \wedge (N \cdot d_V R) = (\dot y\,dx - \dot x\,dy)\,dt,\\
\delta^x \wedge \delta^y \wedge \vartheta_{xy} &\longleftrightarrow \omega_1 = I \circ \pi^{2,1}(\beta^1 \wedge \beta^2 \wedge \beta^3),\\
\gamma_{\mathrm{GF}} &\longleftrightarrow \omega_2 = I \circ \pi^{2,1}(\tau \wedge \alpha \wedge \alpha^1),
\end{aligned}$$

and so on. The form ω_1 is given explicitly by

$$\omega_1 = dt \wedge [\, v\kappa\,dx \wedge dy + \frac{1}{v^2}(\dot y^2 dx \wedge d\dot x - \dot x \dot y(dx \wedge d\dot y + dy \wedge d\dot x) + \dot x^2 dy \wedge d\dot y)\,].$$

We also compute

$$\begin{aligned}
&\pi^{2,1}(\tau \wedge \alpha \wedge \alpha^1)\\
&= \frac{1}{v^4}\,dt \wedge [\,(\dot x \ddot y - \dot y \ddot x)d\dot x \wedge d\dot y + \dot x^2 d\dot x \wedge d\ddot x + \dot x \dot y(d\dot x \wedge d\ddot y + d\dot y \wedge d\ddot x) + \dot y^2 d\dot y \wedge d\ddot y\,].
\end{aligned}$$

This form is d closed and represents the Gelfand-Fuks cohomology of $\mathbf{R}$ in the variational bicomplex. Its canonical representative $I(\omega_2)$ in $\mathcal{F}^3(\mathcal{P}^\infty)$ is quite complicated and need not be given explicitly here.

Finally, we note that the integral of λ around a closed immersed curve in $\mathbf{R}^2$ is, apart from a factor of 2π, the rotation index of the curve.

The computations and results for regular space curves are quite similar. We let $E\colon \mathbf{R} \times \mathbf{R}^3 \to \mathbf{R}$, $G = \operatorname{Diff}^+(\mathbf{R}) \times \operatorname{Eucl}(3)$ and $\mathcal{S}^\infty \subset J^\infty(E)$ be the open set

$$\mathcal{S}^\infty = \{\,(t, R, \dot R, \ddot R, \dddot R, \dots) \in J^\infty(E) \mid \dot R \times \ddot R \neq 0\,\}.$$

The preferred coordinates for $\mathcal{S}^\infty$ are

$$\begin{aligned}
&\mathcal{S}^2\colon (t, R, (T, N, B), v, v', \kappa),\\
&\mathcal{S}^3\colon (t, R, (T, N, B), v, v', \kappa, v'', \kappa', \tau),\\
&\mathcal{S}^4\colon (t, R, (T, N, B), v, v', \kappa, v'', \kappa', \tau, v''', \kappa'', \tau'), \quad \dots
\end{aligned}$$

and so on. Here the Frenet (T, N, B) determines 3 jet coordinates and κ and τ are the curvature and torsion functions. The subbundle $\mathcal{K}^\infty$ is defined by putting the coordinates κ, τ and all their derivatives to zero. It then follows, just as in the case of plane curves, that

$$H^*(\mathcal{E}^*_{\mathrm{pr}\, G}(\mathcal{S}^\infty)) \cong H^*(\mathcal{X}(\mathbf{R})) \otimes H^*(\mathrm{eucl}(3)) \cong \Lambda^*[\gamma_{\mathrm{GF}}, \gamma_1, \gamma_2],$$

where

$$\gamma_1 = \delta^x \wedge \delta^y \wedge \delta^z \quad \text{and} \quad \gamma_2 = \vartheta_{xy} \wedge \vartheta_{yz} \wedge \vartheta_{xz}.$$

The forms γ_1 and γ_2 map to

$$\gamma_1 \longleftrightarrow dx \wedge dy \wedge dz \quad \text{and}$$
$$\gamma_2 \longleftrightarrow \frac{1}{v^2 w^2}((\dot R \times \ddot R) \cdot d\ddot R) \wedge (\dot x d\dot y \wedge d\dot z + \dot y d\dot z \wedge d\dot x + \dot z d\dot x \wedge d\dot y)$$

in the de Rham complex $\Omega^*_{\mathrm{pr}\, G}(\mathcal{S}^\infty)$. Here $w = \|\dot R \times \ddot R\|$.

In [4], we shall use these same techniques to compute the cohomology of the natural Euler-Lagrange complex for immersed n dimensional manifolds in $\mathbf{R}^N$.

§6. The Euler-Lagrange complex for vector field theories on Euclidean space. For this example we let $E = T^*M$ be the cotangent bundle of $M = \mathbf{R}^n$. Our coordinates for E are (x^i, ψ_i). Motivated by vector field theories such as classical electromagnetism, we consider the Lie pseudo-group G generated by the Poincare transformations

$$(A, b) \cdot (x^i, \psi_i) = ((A^{-1})^i_j x^j + b^i, A^i_j \psi_i),$$

where $b = (b^i) \in \mathbf{R}^n$ and $A = (A^i_j) \in \mathrm{SO}(n-1, 1)$, and the gauge transformations

$$\phi \cdot (x^i, \psi_j) = (x^i, \psi_j + \frac{\partial \phi(x^i)}{\partial x^j}),$$

where $\phi(x^i)$ is any real-valued, smooth function on $\mathbf{R}^n$. The Lie algebra Γ of infinitesimal generators for the action of G on T^*M is spanned by the vector fields

$$T_i = \frac{\partial}{\partial x^i}, \quad R_a = -a^j_i x^i \frac{\partial}{\partial x^j} + a^j_i \psi_j \frac{\partial}{\partial \psi_i}, \quad \text{and} \quad X_\phi = \phi_{,i} \frac{\partial}{\partial \psi_i},$$

where the coefficients $(a_{ij}) \in \mathrm{so}(n-1, 1)$. The Lie algebra of formal vector fields Γ_{frm} is spanned by the vector fields T_i, R_a, and

$$X^{j_1 j_2 \cdots j_k} = x^{(j_1} x^{j_2} \cdots x^{j_{k-1}} \partial_\psi^{j_k)}, \quad \text{where} \quad \partial^j_\psi = \frac{\partial}{\partial \psi_j},$$

and where we have used parenthesis to indicate that the enclosed indices are symmetrized. We call the Euler-Lagrange complex $\mathcal{E}^*_{\mathrm{pr}\, G}(J^\infty(T^*M))$ *the Euler-Lagrange complex for vector field theories on* $\mathbf{R}^n$.

The standard d_H homotopy operators (see, for example, [2]) for the free variational bicomplex $\Omega^{*,*}(J^\infty(T^*M))$ are easily seen to be $\operatorname{pr} G$ equivariant and therefore the interior rows of the $\operatorname{pr} G$ invariant variational bicomplex

$$\{\, \Omega^{*,s}_{\operatorname{pr} G}(J^\infty(T^*M)), d_H, I \,\}$$

for $s \geq 1$ are exact. We introduce $\operatorname{pr} G$ adapted coordinates on $J^\infty(T^*M)$ in terms of the field strength tensor or curvature 2 form $F_{ij} = \psi_{j,i} - \psi_{i,j}$ by

$$(x^i,\, \psi_i,\, \psi_{(i,j)},\, F_{ij},\, \psi_{(i,jh)},\, F_{i(j,h)},\, \psi_{(i,jhk)},\, F_{i(j,hk)},\, \dots).$$

The action of $\operatorname{pr} G$ on $J^\infty(T^*M)$ is given in these coordinates by

$$\begin{aligned}&(A,b)\cdot(x^i,\, \psi_i,\, \psi_{(i,j)},\, F_{ij},\, \psi_{(i,jh)},\, F_{i(j,h)},\, \dots)\\ &= ((A^{-1})^i_j x^j + b^j,\, A^r_i \psi_r,\, A^r_i A^s_j \psi_{(r,s)},\, A^r_i A^s_j F_{rs},\, A^r_i A^s_j A^t_h \psi_{(r,st)},\, \dots)\end{aligned}$$

and

$$\begin{aligned}&(\phi)\cdot(x^i,\, \psi_i,\, \psi_{(i,j)},\, F_{ij},\, \psi_{(i,jh)},\, F_{i(j,h)},\, \dots)\\ &\quad = (x^i,\, \psi_i + \phi_{,i},\, \psi_{(i,j)} + \phi_{,ij},\, F_{ij},\, \psi_{(i,jh)} + \phi_{,ijh},\, F_{i(j,h)},\, \dots).\end{aligned}$$

The subbundle

$$\mathcal{K}^\infty = \{\, \sigma \in J^\infty(T^*M) \mid F_{ij}(\sigma) = 0, F_{i(j,h)}(\sigma) = 0, \dots \}$$

is clearly a $\operatorname{pr} G$ strong deformation retract of $J^\infty(T^*M)$ on which $\operatorname{pr} G$ acts transitively. We conclude that

$$H^*(\mathcal{E}^*_{\operatorname{pr} G}(J^\infty(T^*M))) \cong H^*(A^*(\Gamma_{\mathrm{frm}}, \Gamma^0_{\mathrm{frm}}(\sigma_0))). \tag{6.1}$$

At this point it is convenient to suppress the action of the Poincare group and consider, for the moment, the subgroup $\widetilde{G}$ of G consisting of the translations and gauge transformations. The associated formal infinitesimal Lie pseudo-group $\widetilde{\Gamma}_{\mathrm{frm}}$ is generated by the vectors T_i and $X^{j_1 j_2 \cdots j_k}$. In this case the formal linear isotropy subalgebra $\widetilde{\Gamma}^0_{\mathrm{frm}}(\sigma_0) = \{\, 0 \,\}$ and hence

$$H^*(\mathcal{E}^*_{\operatorname{pr} \widetilde{G}}(J^\infty(T^*M))) \cong H^*(A^*(\widetilde{\Gamma}_{\mathrm{frm}})).$$

The Lie algebra cohomology of $\widetilde{\Gamma}_{\mathrm{frm}}$ can be computed in two steps. In the first step we extend the Lie algebra $\widetilde{\Gamma}_{\mathrm{frm}}$ to a Lie algebra $\mathbf{g}$ spanned by the vectors T_i, $X^{j_1 j_2 \cdots j_k}$, and Y. The non-zero bracket relations for $\mathbf{g}$ are

$$[\, T_i, X^j \,] = \delta^j_i Y \quad \text{and} \quad [\, T_i, X^{j_1 j_2 \cdots j_k} \,] = \delta^{(j_1}_i X^{j_2 j_3 \cdots j_k)}, \quad k \geq 2.$$

The motivation behind this construction of $\mathbf{g}$ will become apparent shortly. The cochains τ^i, β and $\alpha_{j_1 j_2 \cdots j_k}$ dual to the vectors T_i, Y and $X^{j_1 j_2 \cdots j_k}$ satisfy

$$d\,\tau^i = 0, \quad d\,\beta = \tau^i \wedge \alpha_i, \quad \text{and} \quad d\,\alpha_{j_1 j_2 \cdots j_k} = \tau^l \wedge \alpha_{l j_1 j_2 \cdots j_k}. \tag{6.2}$$

Since the forms τ^i, $\alpha_{j_1 j_2 \cdots j_k}$ are Y invariant, it follows that

$$A^p(\widetilde{\Gamma}_{\mathrm{frm}}) = A^p(\mathbf{g}, \mathbf{h}), \tag{6.3}$$

where $\mathbf{h}$ is the 1 dimensional subalgebra spanned by Y.

The relative Lie algebra cohomology $H^*(A^*(\mathbf{g}, \mathbf{h}))$ can be computed in terms of the Lie algebra cohomology of $\mathbf{g}$ by examining the Hochschild-Serre spectral sequence relative to $\mathbf{h}$ and using the fact that the bottom row of the E_2 term of this spectral sequence gives the sought after relative cohomology, that is,

$$E_2^{p,0} \cong H^p(A^*(\mathbf{g}, \mathbf{h})).$$

The non-zero terms in the E^0 term of the Hochschild-Serre spectral sequence for $\mathbf{g}$ relative to $\mathbf{h}$ are

β	$\beta \wedge A^1(\mathbf{g}, \mathbf{h})$	$\beta \wedge A^2(\mathbf{g}, \mathbf{h})$	$\beta \wedge A^3(\mathbf{g}, \mathbf{h})$	$\beta \wedge A^4(\mathbf{g}, \mathbf{h})$	$\cdots$
1	$A^1(\mathbf{g}, \mathbf{h})$	$A^2(\mathbf{g}, \mathbf{h})$	$A^3(\mathbf{g}, \mathbf{h})$	$A^4(\mathbf{g}, \mathbf{h})$	$\cdots$

. (6.4)

The d_0 differential is identically zero, so that (6.4) is also the E^1 term. Since $d_1\beta = 0$ and the d_1 differential acting on $A^*(\mathbf{g}, \mathbf{h})$ coincides with the full exterior derivative acting on $A^*(\mathbf{g}, \mathbf{h})$, the E^2 is obviously

β	$\beta \wedge H^1(\mathbf{g}, \mathbf{h})$	$\beta \wedge H^2(\mathbf{g}, \mathbf{h})$	$\beta \wedge H^3(\mathbf{g}, \mathbf{h})$	$\beta \wedge H^4(\mathbf{g}, \mathbf{h})$	$\cdots$
1	$H^1(\mathbf{g}, \mathbf{h})$	$H^2(\mathbf{g}, \mathbf{h})$	$H^3(\mathbf{g}, \mathbf{h})$	$H^4(\mathbf{g}, \mathbf{h})$	$\cdots$

.

Let $c = \tau^i \wedge \alpha_i$. Since $d_2\beta = c$, we can now conclude that

$$\begin{aligned}
H^1(\mathbf{g}, \mathbf{h}) &= H^1(\mathbf{g}), \\
H^2(\mathbf{g}, \mathbf{h}) &= H^2(\mathbf{g}) \oplus \langle c \rangle, \\
H^3(\mathbf{g}, \mathbf{h}) &= H^3(\mathbf{g}) \oplus \big(\langle c \rangle \otimes H^1(\mathbf{g}, \mathbf{h})\big) \\
&= H^3(\mathbf{g}) \oplus \big(\langle c \rangle \otimes H^1(\mathbf{g})\big), \\
H^4(\mathbf{g}, \mathbf{h}) &= H^4(\mathbf{g}) \oplus \big(\langle c \rangle \otimes H^2(\mathbf{g}, \mathbf{h})\big) \\
&= H^4(\mathbf{g}) \oplus \big(\langle c \rangle \otimes H^2(\mathbf{g})\big) \oplus \langle c^2 \rangle
\end{aligned}$$

and, in general, that

$$H^*(A^*(\mathbf{g},\mathbf{h})) = H^*(A^*(\mathbf{g})) \otimes \mathbf{R}[c]. \tag{6.5}$$

It is important to note that $c^{n+1} = (\tau^i \wedge \alpha_i)^{n+1} = 0$.

It therefore remains to determine the Lie algebra cohomology of $\mathbf{g}$. We now consider the spectral sequence relative to the subalgebra generated by the vectors $X^{j_1 j_2 \cdots j_k}$ and Y. The E^0 term of this spectral sequence is

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\Lambda^2[\alpha,\beta]$	$\Lambda^2[\alpha,\beta]\otimes\Lambda^1[\tau]$	$\Lambda^2[\alpha,\beta]\otimes\Lambda^2[\tau]$	$\cdots$	$\Lambda^2[\alpha,\beta]\otimes\Lambda^n[\tau]$
$\Lambda^1[\alpha,\beta]$	$\Lambda^1[\alpha,\beta]\otimes\Lambda^1[\tau]$	$\Lambda^1[\alpha,\beta]\otimes\Lambda^2[\tau]$	$\cdots$	$\Lambda^1[\alpha,\beta]\otimes\Lambda^n[\tau]$
1	$\Lambda^1[\tau]$	$\Lambda^2[\tau]$	$\cdots$	$\Lambda^n[\tau]$

(6.6)

The d_0 differential is zero so that this is also the E_1 term. Since the d_1 structure equations are the same as (6.2), we can identify (6.6) with the constant coefficient variational bicomplex on the trivial bundle $(x^i, u) \to (x^i)$ by identifying τ^i with dx^i, β with the contact form θ, and $\alpha_{j_1 j_2 \cdots j_k}$ with the higher order contact forms $\theta_{j_1 j_2 \cdots j_k}$. It is this identification that motivated the definition of the extension $\mathbf{g}$ in the first place. Since the interior rows of the variational bicomplex are exact, the interior rows of (6.6) are exact and hence

$$H^*(\mathbf{g}) = \Lambda^*[\tau]. \tag{6.7}$$

It may be of some interest to translate the d_H homotopy operator for the variational bicomplex on the trivial bundle $(x^i, u) \to (x^i)$ into our Lie algebra setting. For this we define, for $\omega \in E_1^{p,q}$,

$$F^{i_1 i_2 \cdots i_k}(\omega) = \sum_{l=0}^{\infty} \binom{k+l}{l} (-1)^l \mathcal{L}_{T_{j_1}} \mathcal{L}_{T_{j_2}} \cdots \mathcal{L}_{T_{j_l}} \bigl(X^{i_1 i_2 \cdots i_k j_1 j_2 \cdots j_l} \lrcorner\, \omega \bigr).$$

Then it is easy to see that

$$F^{i_1 i_2 \cdots i_k}(\mathcal{L}_{T_j}\omega) = \delta_j^{(i_1} F^{i_2 \cdots i_k)}(\omega),$$

from which it follows, by direct computation, that the map

$$h_1 \colon E_1^{p,q} \to E_1^{p-1,q}$$

defined for $q \geq 1$ by

$$h_1(\omega) = \frac{1}{q}\sum_{k=0}^{\infty} \frac{k+1}{n-p+k+1}\mathcal{L}_{T_{j_1}}\mathcal{L}_{T_{j_2}}\cdots\mathcal{L}_{T_{j_k}}\left(\alpha \wedge F^{j_1 j_2\cdots j_k l}(T_l \lrcorner\, \omega)\right) \tag{6.8}$$

satisfies

$$\omega = h_1 d_1(\omega) + d_1 h_1(\omega).$$

This proves that the complexes $\{\, E_1^{*,q}, d_1 \,\}$ are exact for $q \geq 1$.

The combination of (6.5) and (6.7) implies that

$$H^*(A^*(\widetilde{\Gamma}_{\text{frm}})) \cong \Lambda^*[\tau] \otimes \mathbf{R}[c].$$

Since the forms dx^i and $d\psi = d\psi_i \wedge dx^i$ are $\widetilde{G}$ invariant, d closed, and satisfy

$$\varrho(dx^i) = \tau^i \quad \text{and} \quad \varrho(d\psi) = c,$$

we deduce immediately that

$$H^*(\Omega^*_{\text{pr}\,\widetilde{G}}(J^\infty(T^*M))) = \Lambda^*[dx^i] \otimes \mathbf{R}[d\psi].$$

Thus a basis for the cohomology of the Euler-Lagrange complex $\mathcal{E}^*_{\text{pr}\,\widetilde{G}}(J^\infty(T^*M))$ is given by the representatives

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge F^l \in \Omega^{r,0}_{\text{pr}\,\widetilde{G}}(J^\infty(T^*M)), \quad \text{where } k + 2l = r, \tag{6.9a}$$

and

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge F^l \wedge (d_V\psi)^s \in \mathcal{F}^s_{\text{pr}\,\widetilde{G}}(J^\infty(T^*M)), \quad \text{where } k + 2l + s = n. \tag{6.9b}$$

We now return to the original invariance group G. The formal infinitesimal isotropy group $\Gamma^0_{\text{frm}}(\sigma_0)$, computed at the point $\sigma_0 \in \mathcal{K}^\infty$ with coordinates

$$x^i = 0, \quad \psi_i = 0, \quad \text{and} \quad \psi_{(i,j_1 j_2\cdots j_k)} = 0$$

is exactly the Lie algebra $\text{so}(n-1,1)$ of the Lorentz group and therefore

$$H^*(\mathcal{E}^*_{\text{pr}\,G}(J^\infty(T^*M))) \cong H^*(A^*(\Gamma_{\text{frm}}, \text{so}(n-1,1))). \tag{6.10}$$

We can proceed just as before, but now with $\mathbf{h} = \langle Y \rangle \oplus \text{so}(n-1,1)$ to deduce that

$$H^*(A^*(\Gamma_{\text{frm}}, \text{so}(n-1,1))) \cong H^*(\mathbf{g}, \text{so}(n-1,1)) \otimes \text{Sym}^*[c].$$

There are no non-zero $\text{so}(n-1,1)$ invariant forms in $\Lambda^p[\tau]$ for $p < n$ and so the bottom row $E_1^{p,0}$ of (6.6) in the relative spectral sequence for $A^*(\mathbf{g}, \text{so}(n-1,1))$ vanishes for $p < n$. The volume form

$$\nu = \tau^1 \wedge \tau^2 \wedge \cdots \wedge \tau^n$$

generates $E_1^{n,0}$. The homotopy operator (6.8) satisfies

$$h_1(\mathcal{L}_{R_a}(\omega)) = \mathcal{L}_{R_a} h_1(\omega)$$

and therefore the interior rows of (6.6) remain exact. The cohomology of the complex $A^*(\mathbf{g}, \mathrm{so}(n-1,1))$ is therefore generated by ν and consequently

$$H^*(A^*(\Gamma_{\mathrm{frm}}, \mathrm{so}(n-1,1))) \cong \Lambda^*[\nu] \otimes \mathbf{R}[c],$$

where $\nu \wedge c = 0$ and $c^{n+1} = 0$. The cohomology of the Euler-Lagrange complex $\mathcal{E}^*_{\mathrm{pr}\,G}(J^\infty(T^*M))$ for vector field theories on $\mathbf{R}^n$ is therefore generated by the forms

$$(d_H\psi)^l = F^l \in \Omega^{2l,0}_{\mathrm{pr}\,G}(J^\infty(T^*M))$$
$$\nu = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \in \Omega^{n,0}_{\mathrm{pr}\,G}(J^\infty(T^*M))$$

and

$$(d_H\psi)^l \wedge (d_V\psi)^s = F^l \wedge (\theta_i \wedge dx^i)^s \in \mathcal{F}^s_{\mathrm{pr}\,G}(J^\infty(T^*M)),$$

where $2l + s = n$. In particular, if $\lambda \in \Omega^{n,0}_{\mathrm{pr}\,G}(J^\infty(T^*M))$ is a null Lagrangian, that is, $E(\lambda) = 0$, then there is a form $\eta \in \Omega^{n-1,0}_{\mathrm{pr}\,G}(J^\infty(T^*M))$ and constants a and b such that

$$\lambda = \begin{cases} b\nu + d_H\eta, & \text{if } n \text{ is odd, and} \\ aF^{n/2} + b\nu + d_H\eta, & \text{if } n \text{ is even.} \end{cases}$$

Similarly, if $\Delta \in \mathcal{F}^1_{\mathrm{pr}\,G}(J^\infty(T^*M))$ is a locally variational source form, that is, Δ satisfies the Helmholtz conditions $\delta_V\Delta = 0$, then there is an invariant Lagrangian $\lambda \in \Omega^{n,0}_{\mathrm{pr}\,G}(J^\infty(T^*M))$ and a constant a such that

$$\Delta = \begin{cases} E(\lambda), & \text{if } n \text{ is even, and} \\ aF^{(n-1)/2} \wedge (dx^i \wedge \theta_i) + E(\lambda), & \text{if } n \text{ is odd.} \end{cases}$$

The source form $F^{(n-1)/2} \wedge (dx^i \wedge \theta_i)$ is called the Chern-Simons mass term for vector field theories.

Naturally, it is of interest to extend this analysis to Yang-Mill field theories on $\mathbf{R}^n$ and for this purpose we would set $E = T^*M \otimes \mathbf{g}$, where $\mathbf{g}$ is a semi-simple Lie algebra. The invariance group for the Euler-Lagrange complex for Yang-Mills theories is the Poincare group, lifted to E together with, of course, the gauge transformations $\psi \to g\psi g^{-1} + g dg^{-1}$. Our general theory applies with the obvious modifications although in this case the final computations of the Lie algebra cohomology $H^*(A^*(\Gamma_{\mathrm{frm}}, \Gamma^0_{\mathrm{frm}}(\sigma_0)))$ are considerably more involved. Details will be presented in a forthcoming paper. It suffices to say that the Lie algebra cohomology $H^*(A^*(\Gamma_{\mathrm{frm}}, \Gamma^0_{\mathrm{frm}}(\sigma_0)))$ is completely determined by the Ad invariant polynomials on $\mathbf{g}$ and therefore our fundamental isomorphism (4.5) for the cohomology of invariant Euler-Lagrange complexes reduces to the well-known Chern-Weil and Chern-Simons constructions.

§7. References.

1. Ian M. Anderson, *Introduction to the variational bicomplex*, Mathematical Aspects of Classical Field Theory, Contemporary Mathematics 132, (M. Gotay, J. Marsden and V. Moncrief, eds.), Amer. Math Soc., Providence, 1992, pp. 51–73.
2. I. M. Anderson, *The Variational Bicomplex*, Academic Press, Boston (to appear).
3. I. M. Anderson and J. Pohjanpelto, *The cohomology of invariant variational bicomplexes*, Acta. Appl. Math. **41** (1995), 3–19.
4. I. M. Anderson and J. Pohjanpelto, *Invariant variational bicomplexes and Lie algebra cohomology*, in preparation.
5. R. Bott, *Notes on Gelfand-Fuks Cohomology and Characteristic Classes*, Raoul Bott: Collected Papers, Vol. 3, Birkhauser, Boston, 1995, pp. 288–356.
6. W. S. Cheung, *C^∞-invariants on loop spaces*, Proc. Amer. Math. Soc. **100** (1987), 322–328.
7. M. Dubois-Violette, M. Henneaux, M. Talon, C.-M. Viallet, *General solution of the consistency equation*, Phys. Lett. B **289** (1992), 361–367.
8. D. B. Fuks, *Cohomology of Infinite Dimensional Lie Algebras*, Consultants Bureau, New York, 1986.
9. V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics*, Cambridge University Press, Cambridge, 1990.
10. M. Henneaux, *Remarks on the renormalization of gauge invariant operators in Yang-Mills theory*, Phys. Lett. B **313** (1993), 35–40.
11. D. J. Saunders, *The Geometry of Jet Bundles*, Cambridge Univ. Press, Cambridge, 1989.
12. T. Tsujishita, *On variation bicomplexes associated to differential equations*, Osaka J. Math. **19** (1982), 311–363.
13. A. M. Vinogradov, *The $\mathcal{C}$-spectral sequence, Lagrangian formalism and conservation laws I, II*, J. Math. Anal. Appl. **100** (1984), 1–129.

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