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EINSTEIN EQUATIONS DO NOT ADMIT A GENERIC POTENTIAL

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Abstract. In 1970, J. Wheeler, at Princeton, offered a 1000$ award to know wether Einstein equations in vacuum should admit or not a generic potential, that is to say if it could be possible to express a solution metric by means of a few arbitrary functions and their derivatives up to a certain order, in such a way that this metric should become a solution of Einstein equations and no others.
Such a problem is a particular case of the following one: how to recognize when a given linear partial differential operator is expressing all the compatibility conditions of a preceding one in a differential sequence.
The purpose of our lecture is to give an *effective answer* to this question by means of the new technique of *differential duality* and to apply it to Einstein equations, thus solving the above challenge negatively.

I/ **THE CHALLENGE** (1970):

The basic motivation of this problem is the search for a Key words: *potential* in order to describe the *field* of a physical theory whenever it satisfies the corresponding *field equations*. The simplest typical example is that of Maxwell equations:

$$dA = F \qquad \Longrightarrow \qquad dF = 0$$

where A is the potential 1-form and F the field 2-form. As a matter of fact, when F is exact, it is also closed. More generally, the idea is to consider the field as the right member of a linear inhomogeneous system where the left member only involves the potential and to consider the field equations just as generating exactly *all* the corresponding compatibility conditions.
In the linear framework, the answer is unique, contrary to what can happen in the nonlinear framework (5).
However, when only the field equations are known, the problem to "*parametrize*" them in a certain sense by means of a potential is much more delicate.

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An unsolved example has been a 1000$ challenge proposed by John Wheeler in 1970 when he was at the physics department of Princeton University, while the author of this paper was a student of Donald C. Spencer at the mathematics department of the same university.
To fix the notations, let $\omega = (+,+,+,-)$ be the Minkowski metric of space-time and Ω be a perturbation of ω as an arbitrary symmetric tensor.
To any metric ω, one can associate the *Christoffel symbols*:

$$\gamma^k_{ij} = \frac{1}{2}\omega^{kr}(\partial_i\omega_{rj} + \partial_j\omega_{ir} - \partial_r\omega_{ij})$$

and the *Riemann tensor*:

$$\rho^k_{lij} = \partial_i\gamma^k_{lj} - \partial_j\gamma^k_{li} + \gamma^r_{lj}\gamma^k_{ri} - \gamma^r_{li}\gamma^k_{rj}$$

from which we may define the *Ricci tensor* $\rho_{ij} = \rho_{ji} = \rho^r_{ijr}$, the *scalar curvature* $\rho = \omega^{ij}\rho_{ij}$ and the *Einstein tensor* $\sigma_{ij} = \rho_{ij} - \frac{1}{2}\omega_{ij}\rho$. By linearization around the Minkowski metric, still denoted by ω, and using capital letters for the variations, we obtain at once:

$$\Gamma^k_{ij} = \frac{1}{2}\omega^{kr}(\partial_i\Omega_{rj} + \partial_j\Omega_{ir} - \partial_r\Omega_{ij})$$

$$R^k_{lij} = \partial_i\Gamma^k_{lj} - \partial_j\Gamma^k_{li}.$$

Setting $\Omega = \omega^{rs}\Omega_{rs}$, we deduce:

$$\Gamma^r_{ri} = \frac{1}{2}\omega^{rs}\partial_i\Omega_{rs} = \frac{1}{2}\partial_i\Omega$$

$$2R_{ij} = \partial_{ij}\Omega + \omega^{rs}\partial_{rs}\Omega_{ij} - \omega^{rs}(\partial_{ri}\Omega_{sj} + \partial_{rj}\Omega_{si})$$

$$R = \omega^{uv}\partial_{uv}\Omega - \omega^{rs}\omega^{uv}\partial_{ru}\Omega_{sv}$$

$$\begin{aligned} 2\Sigma_{ij} = \partial_{ij}\Omega + \omega^{rs}\partial_{rs}\Omega_{ij} \quad & -\omega^{rs}(\partial_{ri}\Omega_{sj} + \partial_{rj}\Omega_{si}) \\ & -\omega_{ij}\omega^{rs}\omega^{uv}(\partial_{uv}\Omega_{rs} - \partial_{ru}\Omega_{sv}) \end{aligned}$$

and we check at once the byproduct of the *Bianchi identities* expressed in the well known (covariant) divergence form:

$$\omega^{ki}\partial_k\Sigma_{ij} \equiv 0$$

If we want to consider Ω as a field and the Einstein equations in vacuum, namely $R_{ij} = 0$ or $\Sigma_{ij} = 0$, as field equations, we are led from the above to the following question:

Question: Can one find certain arbitrary functions ξ and a linear differential operator $\mathcal{D}$ to express Ω in such a way that *all* the compatibility conditions of the linear inhomogeneous system $\mathcal{D}\xi = \Omega$ are generated by $R_{ij} = 0$ or $\Sigma_{ij} = 0$ *only* ?

II/ THE PROBLEM:

The preceding question can be extended to a more general situation as follows.
Let $E, ...$ be vector bundles over a ground differentiable manifold X of dimension n and $\mathcal{D}, ...$ be linear differential operators. We shall denote by the same notation

a bundle and its sheaf of germs of sections as the context will always be clear enough to indicate which interpretation must be taken.
Let us consider a *differential sequence*:

$$E \xrightarrow{\mathcal{D}} F_O \xrightarrow{\mathcal{D}_1} F_1$$

that is to say a situation where $\mathcal{D}_1 \circ \mathcal{D} \equiv 0$ and $\mathcal{D}_1$ generates all the compatibility conditions for $\mathcal{D}$. We shall distinguish between two problems:

DIRECT PROBLEM: Given $\mathcal{D}$, find $\mathcal{D}_1$?

Such a problem has been solved in first place by M. Janet in 1920 (2) and then studied extensively in the period 1965-1975 by D.C. Spencer (6) in USA, with intrinsic tools such as homological algebra, diagram chasing, Spencer cohomology. The latest account can be found in the reference (5).

INVERSE PROBLEM: Given $\mathcal{D}_1$, find $\mathcal{D}$?

Such a problem is of course much more difficult than the direct problem which must be used in order to check the consistency of any possible answer and had never been solved up to now. In case of a positive answer, we shall set:

Definition: $\mathcal{D}_1$ is said to be (formally) *parametrized* by $\mathcal{D}$.

Example: *rot* is parametrized by *grad* while *div* is parametrized by *rot* but the parametrization needs not be unique of course.

III/ **THE FORMAL ADJOINT**:

Surprisingly, the main tool will be exhibited through the use of the "*formal adjoint*" that we now recall.
If $T = T(X)$ is the tangent bundle of X and T^* is the cotangent bundle, we shall denote by $\bigwedge^r T^*$ the bundle of r-forms. Now, if E is a vector bundle over X, we shall denote by E^* its *dual bundle* (inversion of the transition matrix) and by $\tilde{E} = \bigwedge^n T^* \otimes E^*$ its *adjoint bundle*. To any differential operator $\mathcal{D} : E \longrightarrow F$, we shall associate its *formal adjoint* $ad(\mathcal{D}) = \tilde{\mathcal{D}} : \tilde{F} \longrightarrow \tilde{E}$, either by integration by part, setting:

$$\langle \mu, \mathcal{D}\xi \rangle = \langle \tilde{\mathcal{D}}\mu, \xi \rangle + d(\quad)$$

after multiplication by convenient test functions, or, in a symbolic way, by using only three formal rules:
1) ad(matrix) = transposed (matrix).
2) $ad(\partial_i) = -\partial_i$.
3) $ad(\mathcal{P} \circ \mathcal{Q}) = ad(\mathcal{Q}) \circ ad(\mathcal{P})$.
It follows that $ad(ad(\mathcal{D})) = \mathcal{D}$ and that the dual of a differential sequence is again a differential sequence.

IV/ THE MAIN FORMULAS:

We shall generalize for differential operators two basic formulas existing for matrices.

1) It is known from differential algebra (3,4) that a linear (partial) differential ideal is always prime, a fact allowing to define the *differential transcendence degree* ($difftrd$) of the corresponding differential extension. By analogy, we shall denote by $difftrd\mathcal{D}$ the maximum number of unknowns ξ that can be given arbitrarily in the equations $\mathcal{D}\xi = 0$ and set:

$$diffrk\,\mathcal{D} = dim\,E - difftrd\,\mathcal{D}$$

More precisely, if X is "*replaced*" by a differential field K and $\mathcal{D}_1\eta = 0$ generates *all* the compartibility conditions for $\mathcal{D}\xi = \eta$ as before, when ξ is arbitrary, we may consider η as a generic solution of $\mathcal{D}_1\eta = 0$. Accordingly, counting the differential transcendence degrees in the tower of differential extensions:

$$K \subset K\langle\eta\rangle \subset K\langle\xi\rangle$$

we have:

$$difftrd\;K\langle\xi\rangle/K = difftrd\;K\langle\xi\rangle/K\langle\eta\rangle + difftrd\;K\langle\eta\rangle/K$$

and thus:

$$dim\,E = difftrd\,\mathcal{D} + difftrd\,\mathcal{D}_1$$

leading to the first important formula:

$$diffrk\,\mathcal{D} + diffrk\,\mathcal{D}_1 = dim\,F_0$$

2) Let us consider the symbol $\sigma_\chi(\mathcal{D})$ of $\mathcal{D}$ with respect to a covector $\chi \in T^*$ (5,6). Whenever $\mathcal{D}$ of order q is *formally integrable*, that is to say when all the resulting PDE existing at any order $q+r$ can be obtained step by step by differentiating only r times the given ones of order q, then it is known (5) that:

$$diffrk\,\mathcal{D} = max_{\chi\in T^*} rk\;\sigma_\chi(\mathcal{D}).$$

On the other side, one has:

$$ord(\mathcal{D}) = ord(\tilde{\mathcal{D}}) = q \Longrightarrow \sigma_\chi(\tilde{\mathcal{D}}) = (-1)^q\widetilde{\sigma_\chi(\mathcal{D})}$$

Now, if $\mathcal{D}$ is not formally integrable, one can find (1,5) an *injective* operator $\mathcal{P}$ such that $\mathcal{D}' = \mathcal{P}\circ\mathcal{D}$ is formally integrable with the same solutions as $\mathcal{D}$. Accordingly, we have $\tilde{\mathcal{D}}' = \tilde{\mathcal{D}}\circ\tilde{\mathcal{P}}$ with $\tilde{\mathcal{P}}$ *surjective* and thus $\tilde{\mathcal{D}}'$ has the same image as $\tilde{\mathcal{D}}$, leading to $diffrk\;\tilde{\mathcal{D}}' = diffrk\;\tilde{\mathcal{D}}$. From the above symbol property, taking into account that $\tilde{\mathcal{D}}'$ may not be formally integrable even whenever $\mathcal{D}'$ is, we finally have:

$$diffrk\,\tilde{\mathcal{D}} = diffrk\,\tilde{\mathcal{D}}' \geq diffrk\,\mathcal{D}' = diffrk\,\mathcal{D}$$

As $\tilde{\tilde{\mathcal{D}}} = \mathcal{D}$, we deduce the second important formula:

$$diffrk\,\tilde{\mathcal{D}} = diffrk\,\mathcal{D}.$$

V/ **THE ALGORITHM**:

Let us follow successively the next five steps:
1) We start with an operator $\mathcal{D}_1$.
2) We construct the formal adjoint $\tilde{\mathcal{D}}_1$of $\mathcal{D}_1$.
3) We construct all the compatibility conditions $\tilde{\mathcal{D}}$ of $\tilde{\mathcal{D}}_1$.
4) We construct the formal adjoint $\mathcal{D}$ of $\tilde{\mathcal{D}}$.
5) We construct all the compatibility conditions $\mathcal{D}_1{}'$ of $\mathcal{D}$.

As $\tilde{\mathcal{D}} \circ \tilde{\mathcal{D}}_1 \equiv 0$ by assumption, we obtain by duality $\mathcal{D}_1 \circ \mathcal{D} \equiv 0$. Hence, $\mathcal{D}_1$ is only giving a certain number of compatibility conditions for $\mathcal{D}$, a fact simply denoted in a symbolic way by the inequality $\mathcal{D}_1 \leq \mathcal{D}_1{}'$.
Applying twice the first important formula, we get:

$$diffrk\,\mathcal{D} + diffrk\,\mathcal{D}_1{}' = dim\,F_0$$

$$diffrk\,\tilde{\mathcal{D}} + diffrk\,\tilde{\mathcal{D}}_1 = dim\,\tilde{F}_0 = dim\,F_0.$$

Substracting and applying twice the second important formula, we finally get:

$$diffrk\,\mathcal{D}_1 = diffrk\,\mathcal{D}_1{}'.$$

Hence any new compatibility condition brought by $\mathcal{D}_1{}'$ is differentially dependent on the ones already brought by $\mathcal{D}_1$, a fact proving (because a free module is torsion free) that $\mathcal{D}_1$ is parametrizable (by $\mathcal{D}$ indeed) if and only if the equality $\mathcal{D}_1 = \mathcal{D}_1{}'$ holds.

VI/ **THE SOLUTION**:

If $\mathcal{D}_1$ is the linearized Ricci tensor, the preceding algorithm does not give anything simple. *Surprisingly*, on the contrary, if one deals with the linearized Einstein tensor already described in the first section with six terms ordered from 1 to 6 as they follow, one may contract Σ_{ij} with as many test functions λ^{ij} and obtain after two integrations by part:

$$\begin{aligned}\omega^{rs}\Omega_{rs}\partial_{ij}\lambda^{ij} + \omega^{rs}\Omega_{ij}\partial_{rs}\lambda^{ij} \quad & -\omega^{rs}(\Omega_{sj}\partial_{ri}\lambda^{ij} + \Omega_{si}\partial_{rj}\lambda^{ij}) \\ & -\omega_{ij}\omega^{rs}\omega^{uv}(\Omega_{rs}\partial_{uv}\lambda^{ij} - \Omega_{sv}\partial_{ru}\lambda^{ij})\end{aligned}$$

Changing the indices in order to factor out Ω_{ij}, we finally get in succession, with $\lambda = \omega_{ij}\lambda^{ij}$:

$$\omega^{ij}\partial_{rs}\lambda^{rs} + \omega^{rs}\partial_{rs}\lambda^{ij} - \omega^{ri}\partial_{rs}\lambda^{sj} - \omega^{rj}\partial_{rs}\lambda^{si} - \omega^{ij}\omega^{uv}\partial_{uv}\lambda + \omega^{ri}\omega^{sj}\partial_{rs}\lambda$$

that is to say the same terms (with indices upside-down) but now in the respective position 6,2,3,4,5,1, a result proving that *the Einstein operator is self-adjoint*. It follows that $\tilde{\mathcal{D}}$ is the well known divergence already described and $\mathcal{D}$ is therefore the *Killing operator*, that is the Lie derivative of the Minkowski metric with respect to an arbitrry vector field. It is known in this case (5) that $\mathcal{D}_1{}'$ is the linearized Riemann tensor and thus $\mathcal{D}_1 < \mathcal{D}_1{}'$, a result proving that $\mathcal{D}_1$ cannot be parametrized.

As a conclusion, we may say that Einstein equations in vacuum do not admit a generic potential and cannot therefore be considered as "field equations" in the classical Faraday sense, like Maxwell equations.

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