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DISCONTINUOUS SYMMETRIES OF DIFFERENTIAL EQUATIONS

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ABSTRACT. An interaction between discontinuous symmetries and/or solutions and possible meaningful types of breaks is discussed. The general concept is illustrated by a system describing 3+1 magnetohydrodynamics of plasma in tokamak type facilities.

1. General remarks

What are the symmetries of a differential equation good for? Since they are naturally interpreted as (infinitesimal) transformations of solutions of the equation, there are two ways to exploit them. First, one may study the fixed points of such a transformation. Corresponding solutions are called *invariant.* Second, given a symmetry of a differential equation, and its solution, one may generate a 1-parameter family of solutions starting with an initial one. The process is called the *deformation of a solution.*

A natural question is what happens if a solution and/or a symmetry are discontinuous. When deforming a continuous solutions by means of a discontinuous symmetry one generally gets a family of discontinuous solutions (which may be a way of understanding of the development of singularities). And invariant solutions for a discontinuous symmetry are generally discontinuous, too.

The inclusion of discontinuous symmetries into consideration enlarges symmetry algebras for individual differential equations and increases their dimensions (for instance, it is possible to compose a symmetry from two different symmetries on different sides of a break). Yet there is a catch. In case of equations of mathematical physics, their physically meaningful solutions have singularities only on characteristic manifolds. This constraints the possible choice of discontinuity. There are also more delicate restrictions related to Hugoniot-Rankine conditions on breaks.

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2. Terminology

A system of differential equations

$$\mathbf{F}(\mathbf{x}, \mathbf{u}, \mathbf{u_x}, \dots) = 0,$$

where $\mathbf{x}$ is a vector of independent variables, $\mathbf{u}$ is a vector of dependent variables, $\mathbf{u_x}, \dots$, are derivatives and $\mathbf{F}$ is a vector function defines a submanifold in a jet space with coordinates $\mathbf{x}, \mathbf{u}, \mathbf{u_x}, \dots$. A symmetry is a vector field on the jet space which is tangent to this submanifold and such that translation along its trajectories (if any) carry the set of solutions of the system into itself. Any symmetry is uniquely determined by its *generating function,* which, in its turn, defines the above translation (or deformation) by the formula

$$\frac{\partial \mathbf{u}}{\partial \tau} = \mathcal{S} = \begin{pmatrix} S_1(\mathbf{x}, \mathbf{u}, \mathbf{u_x}, \dots) \\ \cdots \\ S_m(\mathbf{x}, \mathbf{u}, \mathbf{u_x}, \dots) \end{pmatrix},$$

where τ is a deformation parameter, $\mathcal{S}$ is the generating function and m is a number of dependent variables. The relation between vector fields and generating functions is as follows. If $X_{\mathcal{S}}$ is a field, then $S_i = \omega_i(X)$, where $\omega_i = du^i - \sum_{j=1}^m u^i_{x_j} dx_j$ are the Cartan forms.

In a special case of *point symmetries* the corresponding vector fields always have trajectories; their generating functions are linear in first order derivatives and do not depend on higher order ones. These vector fields are uniquely determined by their restriction on the (x, u) space.

3. Example from Magnetohydrodynamics

Consider a system

$$\begin{aligned} &\frac{\partial \psi}{\partial t} + [\nabla_\perp \varphi \times \nabla_\perp \Delta_\perp \psi]_z = \frac{\partial \varphi}{\partial z}, \\ &\frac{\partial}{\partial t}\Delta_\perp \varphi + [\nabla_\perp \varphi \times \nabla_\perp \Delta_\perp \varphi]_z = \frac{\partial}{\partial z}\Delta_\perp \psi + [\nabla_\perp \psi \times \nabla_\perp \Delta_\perp \psi]_x, \end{aligned}$$

where $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$, $\nabla_\perp = \{\partial/\partial x, \partial/\partial y\}$, and $[\mathbf{u} \times \mathbf{v}]_z = u_x v_y - u_y v_x$ is the z-component of the vector product $[\mathbf{u} \times \mathbf{v}]$ of vectors $\mathbf{u}$ and $\mathbf{v}$. Here φ and ψ are the potentials of the velocity and the cross-component of the magnetic field. This system describes a high temperature plasma in tokamak-type facilities provided that both the plasma pressure and the cross-component of the magnetic field pressure are much less than the pressure created by the longitudinal component of the magnetic field along the z-axis, and the latter is much greater than the lesser radius of the tokamak torus.

This system has the following, rather rich, point symmetry algebra. Its $\mathbb{R}$-linear basis in the space of generating functions is:

1. $\mathcal{A}_\alpha = \begin{pmatrix} \alpha' \cdot (\varphi + \psi) + \alpha \cdot (\varphi_z + \varphi_t) \\ \alpha' \cdot (\varphi + \psi) + \alpha \cdot (\psi_z + \psi_t) \end{pmatrix}$,
2. $\mathcal{B}_\beta = \begin{pmatrix} \beta' \cdot (\varphi - \psi) + \beta \cdot (\varphi_z - \varphi_t) \\ \beta' \cdot (\varphi - \psi) + \beta \cdot (\psi_z - \psi_t) \end{pmatrix}$,
3. $\mathcal{C}_\alpha = \begin{pmatrix} \alpha' \cdot (x^2 + y^2) + 2\alpha \cdot (y\varphi_x - x\varphi_y) \\ \alpha' \cdot (x^2 + y^2) + 2\alpha \cdot (y\psi_x - x\psi_y) \end{pmatrix}$,
4. $\mathcal{D}_\beta = \begin{pmatrix} -\beta' \cdot (x^2 + y^2) + 2\beta \cdot (y\varphi_x - x\varphi_y) \\ \beta' \cdot (x^2 + y^2) + 2\beta \cdot (y\psi_x - x\psi_y) \end{pmatrix}$,
5. $\mathcal{G}_G = \begin{pmatrix} yG_t + G\varphi_x \\ yG_z + G\psi_x \end{pmatrix}$,
6. $\mathcal{H}_G = \begin{pmatrix} -xG_t + G\varphi_y \\ -xG_z + G\psi_y \end{pmatrix}$,
7. $\mathcal{K}_G = \begin{pmatrix} G_t \\ G_z \end{pmatrix}$,
8. $\mathcal{M} = \begin{pmatrix} x\varphi_x + y\varphi_y + 2z\varphi_z + 2t\varphi_t \\ x\varphi_x + y\varphi_y + 2z\varphi_z + 2t\varphi_t \end{pmatrix}$.

Here $\alpha = \alpha(z+t)$, $\beta = \beta(z-t)$, $G = G(z,t)$ are arbitrary functions and subscripts mean differentiations.

The corresponding symmetry algebra basis in form of vector fields on the space of equation variables $(x, y, z, t, \varphi, \psi)$ is as follows

1. $X_{\mathcal{A}} = \alpha'(\varphi + \psi)\left(\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial\psi}\right) - \alpha\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right)$,
2. $X_{\mathcal{B}} = \beta'(\varphi - \psi)\left(\frac{\partial}{\partial\varphi} - \frac{\partial}{\partial\psi}\right) - \beta\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial t}\right)$,
3. $X_{\mathcal{C}} = \alpha'(x^2 + y^2)\left(\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial\psi}\right) - 2\alpha\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)$,
4. $X_{\mathcal{D}} = \beta'(x^2 + y^2)\left(-\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial\psi}\right) - 2\beta\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)$,
5. $X_{\mathcal{G}} = yG_t\frac{\partial}{\partial\varphi} + yG_z\frac{\partial}{\partial\psi} - G\frac{\partial}{\partial x}$,
6. $X_{\mathcal{H}} = -xG_t\frac{\partial}{\partial\varphi} - xG_z\frac{\partial}{\partial\psi} - G\frac{\partial}{\partial y}$,

7. $X_{\mathcal{K}} = G_t \dfrac{\partial}{\partial t} + G_z \dfrac{\partial}{\partial \psi}$,

8. $X_{\mathcal{M}} = -x \dfrac{\partial}{\partial x} - y \dfrac{\partial}{\partial y} - 2z \dfrac{\partial}{\partial z} - 2t \dfrac{\partial}{\partial t}$.

As it is well known (see [1]), characteristics in magnetohydrodynamics are Alfven waves. Those independent on an individual solution are of the form $z \pm t =$ const. Since arbitrary functions α and β depend exactly on $z \pm t$, their breaks produce discontinuity of symmetries 1–4 of the desired type.

Recall that if $\binom{\varphi_0(x,y,z,t)}{\psi_0(x,y,z,t)}$ is a solution of the equation and $\mathcal{S} = \binom{S(x,y,z,t,\varphi,\psi,\dots)}{T(x,y,z,t,\varphi,\psi,\dots)}$ is a symmetry, the deformation of a solution is in this situation a 1-parameter family $\binom{\varphi(\tau,x,y,z,t)}{\psi(\tau,x,y,z,t)}$, which satisfies the initial value problem

$$\frac{\partial}{\partial \tau}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \begin{pmatrix}S\\ T\end{pmatrix}, \qquad \begin{pmatrix}\varphi\\ \psi\end{pmatrix}\Big|_{\tau=0} = \begin{pmatrix}\varphi_0\\ \psi_0\end{pmatrix}.$$

In particular, solutions of the equation

$$\mathcal{S} = \begin{pmatrix} S\left(x, y, z, t, \varphi, \psi, \varphi_x, \psi_x, \varphi_y, \psi_y, \varphi_z, \psi_z, \varphi_t, \psi_t\right) \\ T\left(x, y, z, t, \varphi, \psi, \varphi_x, \psi_x, \varphi_y, \psi_y, \varphi_z, \psi_z, \varphi_t, \psi_t\right) \end{pmatrix} = 0$$

are $\mathcal{S}$-invariant solutions of the initial equation.

For instance, solutions, invariant with respect to the two-dimensional algebra generated by $\mathcal{A}_\alpha$ and $\mathcal{B}_\beta$ are of the form

$$\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \begin{pmatrix} \left(\alpha\left(z+t\right)\right)^{-1} A\left(x,y\right) + \left(\beta\left(z-t\right)\right)^{-1} B\left(x,y\right) \\ \left(\alpha\left(z+t\right)\right)^{-1} A\left(x,y\right) - \left(\beta\left(z-t\right)\right)^{-1} B\left(x,y\right) \end{pmatrix},$$

where A, B are arbitrary solutions of the system

$$A_x B_y - A_y B_x = 0,$$

$$A_x \Delta B_y - A_y \Delta B_x + B_x \Delta A_y - B_y \Delta A_x = 0.$$

The first equation means $B = F(A)$. In particular, if $B = A$, it follows that $\Delta A = F(A)$. It is clear that singularities of α (as well as its zeros) will, generally, yield singularities of invariant solutions.

As another example, consider solutions invariant with respect to $\mathcal{C}_\alpha$. To obtain them one has to solve the system

$$\alpha' \cdot r^2/2 + \alpha \cdot \varphi_\theta = 0,$$

$$\alpha' \cdot r^2/2 + \alpha \cdot \psi_\theta = 0,$$

where (r, θ) are polar coordinates on (x, y) space. Resulting solutions are of the form

$$\varphi = r^2\theta/2\alpha + A\left(r, z, t\right), \quad \psi = r^2\theta/2\alpha + B\left(r, z, t\right).$$

Here functions A and B must satisfy the system

$$-B_t + A_z = (r\alpha'/2\alpha)\,(A-B)_r\,,$$

$$\nabla_\perp\,(A_t - B_z) = [(r\alpha'/2\alpha)\,\nabla_\perp + 2\alpha'/r\alpha]\,(A-B)_r\,.$$

It is noteworthy that the above solutions are multivalued, since they depend on the angle θ explicitly; any choice of a single-valued branch will lead to a discontinuity.

Now let us take a discontinuous symmetry $\mathcal{A}_\alpha$ for $\alpha = \xi^{-1}$, $\xi \equiv z + t$. In case that $\psi_0 \equiv 0$ we obtain

$$\varphi = \tfrac{1}{2}[1 + |\xi|\,(\tau + \xi^2)^{-1/2}]\varphi_0(x, y, sign\,\xi \cdot \sqrt{\tau + \xi^2}, z - t).$$

Thus, on a singular characteristic $\xi = 0$ for $\tau \neq 0$, a jump of the solution arises, which is proportional to $\varphi_0\,(x, y, \sqrt{\tau}, z - t) - \varphi_0\,(x, y, -\sqrt{\tau}, z - t)$.

Another example is $\psi_0 \equiv 0$, $\alpha = sign\,\xi$. It follows that

$$\varphi = \varphi_0\,(x, y, \xi - 2\tau \cdot sign\,\xi, z - t)$$

We remind that the jump of the potential φ has no physical meaning, but if the initial φ_0 is even in the ξ variable, we obtain a jump of its gradient which is physically sensible.

In both examples the Hugoniot-Rankine condition on break holds for the total energy conservation law. That is, there is no jump of the energy $\frac{1}{2}(\varphi_x^2+\varphi_y^2+\psi_x^2+\psi_y^2)$ on characteristics $z + t = \text{const}$ provided the initial function $\varphi_0\,(x, y, \xi, z - t)$ is even in the $\xi = z + t$ variable.

References

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