DIFFERENTIAL GEOMETRY AND APPLICATIONS
Proc. Conf., Aug. 28 – Sept. 1, 1995, Brno, Czech Republic
Masaryk University, Brno 1996, 469–478

A REPRESENTATION OF THE VARIATIONAL SEQUENCE IN HIGHER ORDER MECHANICS

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ABSTRACT. Let Y be a fibered manifold over a base manifold X. A differential form ρ defined on the r-jet prolongation J^rY of Y is said to be *contact* if it vanishes along the r-jet prolongation $J^r\gamma$ of every section γ of Y. The contact forms define a subsequence of the de Rham sequence on J^rY. The corresponding quotient sequence is known as the *r-th order variational sequence*. We consider the case of 1-dimensional base X (*mechanics*).

We construct an isomorphism between the variational sequence and another sequence whose elements are spaces of forms rather than spaces of classes of forms. In this way we determine explicitly the first two mappings of the variational sequence.

1. Introduction

The subject of this paper is the variational sequence of a fibered manifold which was defined by Krupka in [3], especially the case of mechanics i.e. the case of fibered manifolds with 1-dimensional base manifold. Our aim is to give a description of such variational sequence in some form convenient for practical computation.

The variational sequence is defined as the factor complex Ω/Θ of the de Rham complex Ω, where Θ is a certain subcomplex of Ω, so it is natural to try to find some mapping $\Phi : \Omega \to A$ such that $\operatorname{Ker}\Phi = \Theta$ and to express Ω/Θ with help of the corresponding isomorphism $\Omega/\Theta \to \operatorname{Im}\Phi$. Actually, in Definition 1 we construct such a mapping Φ which assigns to every class from Ω/Θ a differential form and in Theorem 1 we prove that $\operatorname{Ker}\Phi = \Theta$. Finally, in Theorem 2 we give the explicit coordinate description of the first three elements of the variational sequence in our representation as well as the coordinate form of first three mappings of Φ. This description enables us to see the connection between the first two mappings of the variational sequence and the Euler–Lagrange and the Helmholtz–Sonin mapping.

Let us note that the problem of the representation of the variational sequence has been solved by Krupka in [4] for the first order mechanics and by Musilová in [2]

1991 *Mathematics Subject Classification.* 49F05, 58A15, 58A20, 58E99.
Key words and phrases. Fibered manifold, jet, contact form, variational sequence.
This paper is in final form and no version of it will be submitted for publication elsewhere.
Research supported by grant No. 201/93/2245 from the Czech Grant Agency.

for the first three elements of the variational sequence of general order. The results of this article coincide with those from [2] and [3] for the first two members of the variational sequence. For the third member Ω^3/Θ^3 of the variational sequence our representation is different, but the corresponding systems of equations (Helmholtz–Sonin equations) are equivalent.

2. Notation

Through the paper we use the same notation as in [2, 3]. Let us briefly recall the main facts. $\pi : Y \to X$ denotes the fibered manifold $pr_1 : \mathbb{R}^{m+1} \to \mathbb{R}$, where pr_1 is the standard projection onto the first coordinate. The canonical coordinates on Y are denoted by $(t, q^1, \dots, q^m)$, the corresponding coordinates on the r-jet prolongation J^rY of Y are denoted by $(t, q^{\sigma_0}, q_1^{\sigma_1}, \dots, q_r^{\sigma_r})$. The canonical jet projection of J^rY onto J^sY, where $0 \le s \le r$, (resp. onto X) is denoted by $\pi_{r,s}$ (resp. π_r). The formal derivative of a function $f \in C^\infty(J^rY)$ in the canonical coordinate system $(t, q^1, \dots, q^m)$ is denoted by $d_1 f$. By $d_k f$ we denote the k-th derivative $d_1(d_1(\dots d_1 f))$ of f. The exterior derivative is denoted by d. The set of all multiindices of the length k over a set A is denoted by $\mathrm{multi}_k(A)$. For a system (F^I_Σ) where $I = (i_1, \dots, i_k) \in \mathrm{multi}_k(A)$, $\Sigma = (\sigma_1, \dots, \sigma_k) \in \mathrm{multi}_k(B)$, we denote by $(F^I_\Sigma)_{\mathrm{anti}}$ the antisymmetrisation of (F^I_Σ) with respect to the indices $\binom{i_1}{\sigma_1}, \dots, \binom{i_k}{\sigma_k}$. By ω_i^σ we denote the contact form $\omega_i^\sigma = d\,q_i^\sigma - q_{i+1}^\sigma\,dt$. For $I = (i_1, \dots, i_k) \in \mathrm{multi}_k(0, \dots, r)$, $\Sigma = (\sigma_1, \dots, \sigma_k) \in \mathrm{multi}_k(0, \dots, m)$ we define $\omega_I^\Sigma = \omega_{i_1}^{\sigma_1} \wedge \dots \wedge \omega_{i_k}^{\sigma_k}$. Whenever we use the coordinate representation $\rho = A^I_\Sigma\,\omega^\Sigma_I$, $\rho = A^I_\Sigma\,dt \wedge \omega^\Sigma_I$ or $\rho = A^{i_1,\dots,i_k}_{\sigma_1,\dots,\sigma_k}\,dq_{i_1}^{\sigma_1} \wedge \dots \wedge dq_{i_k}^{\sigma_k}$ (with the usual Einstein summation) we suppose the antisymmetry of the system $(A^I_\Sigma)_{\substack{I\\ \Sigma}}$ with respect to the indices $\binom{i_1}{\sigma_1}, \dots, \binom{i_k}{\sigma_k}$. By $\mathfrak{J}$, $\mathfrak{Z}$, etc., we denote multiindices of finite length with indices of the form $\mathfrak{J}_i = (0, \dots, 0, 1, 0, \dots, 0)$. (All indices $\mathfrak{J}_i$ of such multiindex $\mathfrak{J}$ are supposed to be of the same length.) If $I \in \mathrm{multi}_k(\mathbb{Z})$ and $\mathfrak{J} = (\mathfrak{J}_1, \dots, \mathfrak{J}_p)$, where $\mathfrak{J}_i = (0, \dots, 1, \dots, 0) \in \mathrm{multi}_k(\{0, 1\})$, then we define $|\mathfrak{J}| = p$, $I - \mathfrak{J} = I - \mathfrak{J}_1 - \dots - \mathfrak{J}_p$. By $A^{i_1,\dots,\overset{p}{i_p},\dots,i_k}$ we denote the fact that i_p is the p-th index of the multiindex $(i_1, \dots, i_k)$.

3. A representation of the variational sequence by forms

Definition 1. Let $k > 0$, let $\eta \in \Omega_r^{k+1}$, let $p_k\eta = \sum_{I \in \mathrm{multi}_k(\mathbb{Z})} A^I_\Sigma dt \wedge \omega^\Sigma_I$, where $A^I_\Sigma = 0$ for $I \notin \mathrm{multi}_k(0, ..., r)$. We denote

$$ {}^\eta F_\Sigma^{(\alpha,K)} = \sum_{|\mathfrak{J}| \ge 0} (-1)^{|\mathfrak{J}|} \sum_{\lambda=0}^{\infty} (-1)^\lambda \binom{\lambda + |\mathfrak{J}|}{|\mathfrak{J}|}\; d_\lambda A_\Sigma^{(\alpha+1+|\mathfrak{J}|+\lambda, K-\mathfrak{J})} \tag{1} $$

for $(\alpha, K) \in \mathrm{multi}_k(\mathbb{Z})$, $\Sigma \in \mathrm{multi}_k(1, ..., m)$,

$$ {}^\eta G^I_\Sigma = ({}^\eta F^I_\Sigma)_{\mathrm{anti}} \tag{2} $$

for $I \in \operatorname{multi}_k(\mathbb{Z})$, $\Sigma \in \operatorname{multi}_k(1, \dots, m)$,

$$\Psi_r^{k+1}(\eta) = \sum_{I \in \operatorname{multi}_k(0,\dots,r)} {}^{\eta}G_{\Sigma}^{I}\, \omega_I^{\Sigma}, \tag{3}$$

$$\Phi_r^{k+1}(\eta) = p_k(\eta - d\,\Psi_r^{k+1}(\eta)). \tag{4}$$

For $k = 0$ we set $\Phi_r^1 = p_0 : \Omega_r^1 \to \Omega_{r+1}^1$.

Remark. (1) All the sums in Definition 1 are in actual fact finite. (2) The formulas in Definition 1 define linear mappings

$$\Phi_r^{k+1} : \Omega_r^{k+1} \to \Omega_{2r+2}^{k+1}, \qquad \Psi_r^{k+1} : \Omega_r^{k+1} \to \Omega_{2r+1\,(c)}^{k}.$$

Lemma 1. *Let* $\rho = \sum_{I \in \operatorname{multi}_k(\mathbb{Z})} H_\Sigma^I\, \omega_I^\Sigma \in \Omega_{s\,(c)}^k$, *where* $H_\Sigma^I = 0$ *for* $I \notin \operatorname{multi}_k(0, \dots, r)$. *Then*

$$p_k d\rho = \sum_{I \in \operatorname{multi}_k(\mathbb{Z})} \Big(d_1 H_\Sigma^I + \sum_{|\mathfrak{J}|=1} H_\Sigma^{I-\mathfrak{J}} \Big)\, dt \wedge \omega_I^\Sigma. \tag{5}$$

Proof. In the usual notation we can write

$$\rho = \sum_{0 \le l_1, \dots, l_k \le r} H_{\sigma_1, \dots, \sigma_k}^{l_1, \dots, l_k}\, \omega_{l_1}^{\sigma_1} \wedge \dots \wedge \omega_{l_k}^{\sigma_k}.$$

Then

$$\begin{aligned}
d\rho = {} & \sum_{0 \le l_1, \dots, l_k \le r} \frac{\partial H_{\sigma_1, \dots, \sigma_k}^{l_1, \dots, l_k}}{\partial t}\, dt \wedge \omega_{l_1}^{\sigma_1} \wedge \dots \wedge \omega_{l_k}^{\sigma_k} \\
& + \sum_{\substack{0 \le l \le s \\ 0 \le l_1, \dots, l_k \le r}} \frac{\partial H_{\sigma_1, \dots, \sigma_k}^{l_1, \dots, l_k}}{\partial q_l^\sigma}\, dq_l^\sigma \wedge \omega_{l_1}^{\sigma_1} \wedge \dots \wedge \omega_{l_k}^{\sigma_k} \\
& + \sum_{\substack{1 \le i \le k \\ 0 \le l_1, \dots, l_k \le r}} (-1)^{i-1}\, H_{\sigma_1, \dots, \sigma_k}^{l_1, \dots, l_k}\, \omega_{l_1}^{\sigma_1} \wedge \dots \wedge d\omega_{l_i}^{\sigma_i} \wedge \dots \wedge \omega_{l_k}^{\sigma_k}, \\
p_k d\rho = {} & \sum_{0 \le l_1, \dots, l_k \le r} \frac{\partial H_{\sigma_1, \dots, \sigma_k}^{l_1, \dots, l_k}}{\partial t}\, dt \wedge \omega_{l_1}^{\sigma_1} \wedge \dots \wedge \omega_{l_k}^{\sigma_k} \\
& + \sum_{\substack{0 \le l \le s \\ 0 \le l_1, \dots, l_k \le r}} q_{l+1}^\tau \frac{\partial H_{\sigma_1, \dots, \sigma_k}^{l_1, \dots, l_k}}{\partial q_l^\tau}\, dt \wedge \omega_{l_1}^{\sigma_1} \wedge \dots \wedge \omega_{l_k}^{\sigma_k} \\
& + \sum_{\substack{1 \le i \le k \\ 0 \le l_1, \dots, l_k \le r}} (-1)^i\, H_{\sigma_1, \dots, \sigma_k}^{l_1, \dots, l_k}\, \omega_{l_1}^{\sigma_1} \wedge \dots \wedge \omega_{l_i+1}^{\sigma_i} \wedge dt \wedge \dots \wedge \omega_{l_k}^{\sigma_k} \\
= {} & \sum_{I \in \operatorname{multi}_k(0, \dots, r)} d_1 H_\Sigma^I\, dt \wedge \omega_I^\Sigma \\
& + \sum_{\substack{1 \le i \le k \\ 0 \le l_1, \dots, l_k \le r}} H_{\sigma_1, \dots, \sigma_k}^{l_1, \dots, l_k}\, dt \wedge \omega_{l_1}^{\sigma_1} \wedge \dots \wedge \omega_{l_i+1}^{\sigma_i} \wedge \dots \wedge \omega_{l_k}^{\sigma_k}.
\end{aligned}$$

Now, we can use the assumption $H^I_\Sigma = 0$ for $I \notin \mathrm{multi}_k\,(0, ..., r)$ and write

$$
\begin{aligned}
p_k d\rho = & \sum_{I \in \mathrm{multi}_k(\mathbb{Z})} d_1 H^I_\Sigma \, dt \wedge \omega^\Sigma_I \\
& + \sum_{\substack{1 \le i \le k \\ 0 \le l_1, \dots, l_k \in \mathbb{Z}}} H^{l_1, \dots, l_i - 1, \dots, l_k}_{\sigma_1, \dots, \sigma_k} \, dt \wedge \omega^{\sigma_1}_{l_1} \wedge \ldots \wedge \omega^{\sigma_k}_{l_k} \\
= & \sum_{I \in \mathrm{multi}_k(\mathbb{Z})} \Big(d_1 H^I_\Sigma + \sum_{|\mathfrak{J}|=1} H^{I-\mathfrak{J}}_\Sigma \Big) \, dt \wedge \omega^\Sigma_I. \qquad \square
\end{aligned}
$$

Lemma 2. *Let* $\left(H^I_\Sigma\right)_{I, \Sigma \in \mathrm{multi}_k(\mathbb{Z})}$ *be an arbitrary system of functions. Then*

$$
\Big(\sum_{|\mathfrak{J}|=1} H^{I-\mathfrak{J}}_\Sigma \Big)_{\mathrm{anti}} = \sum_{|\mathfrak{J}|=1} \Big(H^{I-\mathfrak{J}}_\Sigma \Big)_{\mathrm{anti}}
$$

for each $I, \Sigma \in \mathrm{multi}_k(\mathbb{Z})$.

Remark. The immediate consequence of Lemma 2 is that the expression (5) is coordinate representation of $p_k d\rho$ with antisymmetric coefficients. (The antisymmetry of the coefficients in the representation $\rho = H^I_\Sigma \omega^\Sigma_I$ is guaranteed by our general convention.)

Lemma 3. *Let* $\left({}^\eta G^I_\Sigma\right)_{I \in \mathrm{multi}_k(\mathbb{Z}), \Sigma \in \mathrm{multi}_k(1, \dots, m)}$ *be the system (2) from Definition 1 Then*

$$
d_1 {}^\eta G^I_\Sigma + \sum_{|\mathfrak{J}|=1} {}^\eta G^{I-\mathfrak{J}}_\Sigma = A^I_\Sigma
$$

for each $I \in \mathrm{multi}_k\,(0, \dots, r)$, $\Sigma \in \mathrm{multi}_k\,(1, \dots, m)$.

Proof. Let $I = (i_1, K) \in \mathrm{multi}_k\,(0, \dots, r)$, $\Sigma \in \mathrm{multi}_k\,(1, \dots, m)$. Then

$$
\begin{aligned}
d_1 {}^\eta G^I_\Sigma + \sum_{|\mathfrak{J}|=1} {}^\eta G^{I-\mathfrak{J}}_\Sigma = & \, d_1 \left({}^\eta F^I_\Sigma\right)_{\mathrm{anti}} + \sum_{|\mathfrak{J}|=1} \Big({}^\eta F^{I-\mathfrak{J}}_\Sigma\Big)_{\mathrm{anti}} \\
= & \Big(d_1 {}^\eta F^I_\Sigma + \sum_{|\mathfrak{J}|=1} {}^\eta F^{I-\mathfrak{J}}_\Sigma \Big)_{\mathrm{anti}} \\
= & \Big(\sum_{|\mathfrak{J}| \ge 0} (-1)^{|\mathfrak{J}|} \sum_{\lambda=0}^{\infty} (-1)^\lambda \binom{\lambda + |\mathfrak{J}|}{|\mathfrak{J}|} d_{\lambda+1} A^{i_1+1+|\mathfrak{J}|+\lambda, K-\mathfrak{J}}_\Sigma \\
& + \sum_{|\mathfrak{J}| \ge 0} (-1)^{|\mathfrak{J}|} \sum_{\lambda=0}^{\infty} (-1)^\lambda \binom{\lambda + |\mathfrak{J}|}{|\mathfrak{J}|} d_\lambda A^{i_1+|\mathfrak{J}|+\lambda, K-\mathfrak{J}}_\Sigma \\
& + \sum_{|\mathfrak{L}|=1} \sum_{|\mathfrak{J}| \ge 0} (-1)^{|\mathfrak{J}|} \sum_{\lambda=0}^{\infty} (-1)^\lambda \binom{\lambda + |\mathfrak{J}|}{|\mathfrak{J}|} d_\lambda A^{i_1+1+|\mathfrak{J}|+\lambda, (K-\mathfrak{L})-\mathfrak{J}}_\Sigma \Big)_{\mathrm{anti}}
\end{aligned}
$$

$$
\begin{aligned}
&= \Bigg(- \sum_{|\mathfrak{J}|\geq 0} (-1)^{|\mathfrak{J}|} \sum_{\lambda=1}^{\infty} (-1)^{\lambda} \binom{\lambda + |\mathfrak{J}| - 1}{|\mathfrak{J}|} d_\lambda A_{\Sigma}^{i_1+|\mathfrak{J}|+\lambda, K-\mathfrak{J}} \\
&\quad + \sum_{|\mathfrak{J}|\geq 0} (-1)^{|\mathfrak{J}|} \sum_{\lambda=0}^{\infty} (-1)^{\lambda} \binom{\lambda + |\mathfrak{J}|}{|\mathfrak{J}|} d_\lambda A_{\Sigma}^{i_1+|\mathfrak{J}|+\lambda, K-\mathfrak{J}} \\
&\quad - \sum_{|\mathfrak{J}|\geq 1} (-1)^{|\mathfrak{J}|} \sum_{\lambda=0}^{\infty} (-1)^{\lambda} \binom{\lambda + |\mathfrak{J}| - 1}{|\mathfrak{J}| - 1} d_\lambda A_{\Sigma}^{i_1+|\mathfrak{J}|+\lambda, K-\mathfrak{J}} \Bigg)_{\mathrm{anti}} \\
&= \Bigg((-1)^0 (-1)^0 \binom{0+0}{0} d_0 A_{\Sigma}^{i_1+0+0, K-0} \\
&\quad + (-1)^0 \sum_{\lambda=1}^{\infty} (-1)^{\lambda} \binom{l+0}{0} d_\lambda A_{\Sigma}^{i_1+0+\lambda, K-0} \\
&\quad - (-1)^0 \sum_{\lambda=1}^{\infty} (-1)^{\lambda} \binom{\lambda+0-1}{0} d_\lambda A_{\Sigma}^{i_1+0+\lambda, K-0} \\
&\quad + \sum_{|\mathfrak{J}|\geq 1} (-1)^{|\mathfrak{J}|} (-1)^0 \binom{0+|\mathfrak{J}|}{|\mathfrak{J}|} d_0 A_{\Sigma}^{i_1+|\mathfrak{J}|+0, K-\mathfrak{J}} \\
&\quad - \sum_{|\mathfrak{J}|\geq 1} (-1)^{|\mathfrak{J}|} (-1)^0 \binom{0+|\mathfrak{J}|-1}{|\mathfrak{J}|-1} d_0 A_{\Sigma}^{i_1+|\mathfrak{J}|+0, K-\mathfrak{J}} \\
&\quad + \sum_{|\mathfrak{J}|\geq 1} (-1)^{|\mathfrak{J}|} \sum_{\lambda=1}^{\infty} (-1)^{\lambda} \Bigg[\binom{\lambda+|\mathfrak{J}|}{|\mathfrak{J}|} - \binom{\lambda+|\mathfrak{J}|-1}{|\mathfrak{J}|} \\
&\qquad - \binom{\lambda+|\mathfrak{J}|-1}{|\mathfrak{J}|-1} \Bigg] d_\lambda A_{\Sigma}^{i_1+|\mathfrak{J}|+\lambda, K-\mathfrak{J}} \Bigg)_{\mathrm{anti}} \\
&= \left(A_{\Sigma}^{I}\right)_{\mathrm{anti}} = A_{\Sigma}^{I}.
\end{aligned}
$$

(Due to our convention the system $\left(A_{\Sigma}^{I}\right)$ is antisymmetric.) □

Lemma 4. *Let* $\eta \in \Omega_r^{k+1}$. *Then*

$$
\begin{aligned}
\Phi_r^{k+1}(\eta) = \sum_{p=1,\dots,k} \; \sum_{0\leq a_1<\dots<a_p\leq k} \; \sum_{\substack{I=(i_1,\dots,i_k)\in \mathrm{multi}_k(0,\dots,r) \\ i_{a_1}=\dots=i_{a_p}=0 \\ i_a\neq 0 \text{ for } a\notin\{a_1,\dots,a_p\}}} \\
\left({}^{\eta}G_{\Sigma}^{i_1,\dots,\overset{a_1}{\overline{-1}},\dots,i_k} + \dots + {}^{\eta}G_{\Sigma}^{i_1,\dots,\overset{a_p}{\overline{-1}},\dots,i_k} \right) dt \wedge \omega_I^{\Sigma} \\
- \sum_{0\leq a_1\leq k} \; \sum_{\substack{I\in \mathrm{multi}_k(0,\dots,r+1)\setminus \mathrm{multi}_k(0,\dots,r) \\ i_{a_1}=r+1 \\ i_a\neq r+1 \text{ for } a\neq a_1}} {}^{\eta}G_{\Sigma}^{i_1,\dots,\overset{a_1}{\overline{r}},\dots,i_k} \, dt \wedge \omega_I^{\Sigma}.
\end{aligned}
$$

Proof. Let $p_k\eta = \sum_{I\in \mathrm{multi}_k(\mathbb{Z})} A^I_\Sigma \, dt \wedge \omega^\Sigma_I$, where $A^I_\Sigma = 0$ for $I \notin \mathrm{multi}_k\,(0,\dots,r)$. Let us denote $H^I_\Sigma = {}^\eta G^I_\Sigma$ for $I \in \mathrm{multi}_k\,(0,\dots,r)$ and put $H^I_\Sigma = 0$ for $I \in \mathrm{multi}_k\,(\mathbb{Z}) \setminus \mathrm{multi}_k\,(0,\dots,r)$. $\Psi^{k+1}_r(\eta) = \sum_{I\in \mathrm{multi}_k(\mathbb{Z})} H^I_\Sigma \omega^\Sigma_I$. According to Lemma 1 we can write

$$\Phi^{k+1}_r(\eta) = \sum_{I\in \mathrm{multi}_k(\mathbb{Z})} \Big(A^I_\Sigma - d_1 H^I_\Sigma - \sum_{|\mathfrak{I}|=1} H^{I-\mathfrak{I}}_\Sigma \Big) \, dt \wedge \omega^\Sigma_I .$$

First let us suppose that $I \in \mathrm{multi}_k\,(0,\dots,r)$, $\Sigma \in \mathrm{multi}_k\,(1,\dots,m)$. Let $I = (i_1,\dots,i_k)$, where $i_{a_1} = \dots = i_{a_p} = 0$, $i_a \neq 0$ for $a \notin \{a_1,\dots,a_p\}$. Then

$$\begin{aligned} A^I_\Sigma - d_1 H^I_\Sigma - \sum_{|\mathfrak{I}|=1} H^{I-\mathfrak{I}}_\Sigma &= A^I_\Sigma - d_1 {}^\eta G^I_\Sigma - \sum_{\substack{|\mathfrak{I}|=1 \\ \mathfrak{I}_{a_1}=\dots=\mathfrak{I}_{a_p}=0}} {}^\eta G^{I-\mathfrak{I}}_\Sigma \\ &= A^I_\Sigma - \Big(d_1 {}^\eta G^I_\Sigma + \sum_{|\mathfrak{I}|=1} {}^\eta G^{I-\mathfrak{I}}_\Sigma \Big) + \sum_{l=1\dots p} G^{i_1,\dots,\overset{a_l}{-1},\dots,i_k}_{\sigma_1,\dots,\sigma_k} \\ &= \sum_{l=1\dots p} G^{i_1,\dots,\overset{a_l}{-1},\dots,i_k}_{\sigma_1,\dots,\sigma_k} . \end{aligned}$$

(The last step follows from Lemma 3.)

Now, let us suppose that $I \in \mathrm{multi}_k\,(0,\dots,r+1) \setminus \mathrm{multi}_k\,(0,\dots,r)$ and that $\Sigma \in \mathrm{multi}_k\,(1,\dots,m)$. Let $I = (i_1,\dots,i_k)$, let $i_{a_1} = \dots = i_{a_p} = r+1$, let $i_a \neq r+1$ for $a \notin \{a_1,\dots,a_p\}$. If $p > 1$ then each upper multiindex in the expression $A^I_\Sigma - d_1 H^I_\Sigma - \sum_{|\mathfrak{I}|=1} H^{I-\mathfrak{I}}_\Sigma$ contains $r+1$ which implies $A^I_\Sigma - d_1 H^I_\Sigma - \sum_{|\mathfrak{I}|=1} H^{I-\mathfrak{I}}_\Sigma = 0$.

Let $p = 1$. Then

$$\begin{aligned} A^I_\Sigma - d_1 H^I_\Sigma - \sum_{|\mathfrak{I}|=1} H^{I-\mathfrak{I}}_\Sigma &= -H^{i_1,\dots,i_{a_p}-1,\dots,i_k}_{\sigma_1,\dots,\sigma_k} = -H^{i_1,\dots,\overset{a_p}{r},\dots,i_k}_{\sigma_1,\dots,\sigma_k} \\ &= -{}^\eta G^{i_1,\dots,\overset{a_p}{r},\dots,i_k}_{\sigma_1,\dots,\sigma_k} . \end{aligned}$$

Finally, for $I \in \mathrm{multi}_k\,(\mathbb{Z}) \setminus \mathrm{multi}_k\,(0,\dots,r+1)$, $\Sigma \in \mathrm{multi}_k\,(1,\dots,m)$ we have $A^I_\Sigma - d_1 H^I_\Sigma - \sum_{|\mathfrak{I}|=1} H^{I-\mathfrak{I}}_\Sigma = 0$ because none of the upper multiindices in this expression belongs to $\mathrm{multi}_k\,(0,\dots,r)$. The combination of these results gives us the desired formula. □

Corollary 1. $\mathrm{Im}\,\Phi^{k+1}_r \subseteq \Omega^{k+1}_{2r+1}$ *for each* $k > 0$.

Theorem 1. Ker $\Phi^k_r = \Theta^k_r$ *for each* $k > 0$.

Proof. For $k = 1$ the relation follows directly from the definition of Φ^1_r and Θ^1_r. Now, let us prove that Ker $\Phi^{k+1}_r = \Theta^{k+1}_r$ for each $k > 0$. Let $k > 0$, let $\eta \in \Theta^{k+1}_r$ be an arbitrary form. Let us prove that $\eta \in$ Ker Φ^{k+1}_r. $\eta \in \Theta^{k+1}_r$ implies $\eta = d\alpha + \beta$

for some forms $\alpha \in \Omega^k_{r(c)}$, $\beta \in \Omega^{k+1}_{r(c)}$. Let $\alpha = \sum_{I \in \mathrm{multi}_k(\mathbb{Z})} H^I_\Sigma \, \omega^\Sigma_I$, where $H^I_\Sigma = 0$ for $I \notin \mathrm{multi}_k\,(0, \ldots, r-1)$. According to Lemma 1

$$p_k \eta = p_k d\alpha = \sum_{I \in \mathrm{multi}_k(\mathbb{Z})} \Big(d_1 H^I_\Sigma + \sum_{|\mathfrak{J}|=1} H^{I-\mathfrak{J}}_\Sigma \Big) \, dt \wedge \omega^\Sigma_I. \tag{6}$$

(Let us notice that $d_1 H^I_\Sigma + \sum_{|\mathfrak{J}|=1} H^{I-\mathfrak{J}}_\Sigma = 0$ for $I \notin \mathrm{multi}_k\,(0, \ldots, r)$ as required in Definition 1 and that the expression (6) is coordinate representation of $p_k\eta$ with antisymmetric coefficients.) Then for each $I = (i_1, K) \in \mathrm{multi}_k\,(\mathbb{Z})$, $\Sigma \in \mathrm{multi}_k\,(1, \ldots, m)$ we have

$$\begin{aligned} {}^\eta F^I_\Sigma = \sum_{|\mathfrak{J}| \geq 0} (-1)^{|\mathfrak{J}|} \sum_{\lambda=0}^{\infty} (-1)^\lambda \binom{l + |\mathfrak{J}|}{|\mathfrak{J}|} \; d_\lambda \Big(d_1 H^{i_1+1+|\mathfrak{J}|+\lambda, K-\mathfrak{J}}_\Sigma \\ + H^{i_1+|\mathfrak{J}|+\lambda, K-\mathfrak{J}}_\Sigma + \sum_{|\mathfrak{Z}|=1} H^{i_1+1+|\mathfrak{J}|+\lambda, K-\mathfrak{Z}-\mathfrak{J}}_\Sigma \Big) = H^I_\Sigma. \end{aligned}$$

(The computation is similar to that one in Lemma 3.) Now, according to Lemma 4 we get $\Phi^{k+1}_r(\eta) = 0$ because ${}^\eta G^I_\Sigma = \big({}^\eta F^I_\Sigma\big)_{\mathrm{anti}} = \big(H^I_\Sigma\big)_{\mathrm{anti}} = 0$ for each $I \notin \mathrm{multi}_k(0, \ldots, r-1)$.

It remains to prove the inclusion Ker $\Phi^{k+1}_r \subseteq \Theta^{k+1}_r$. Let $\eta \in$ Ker Φ^{k+1}_r. Then $p_k\,\big(\eta - d\,\Psi^{k+1}_r(\eta)\big) = 0$ which means that both the forms $\Psi^{k+1}_r(\eta)$, $\eta - d\,\Psi^{k+1}_r(\eta)$ are strongly contact. If we prove that these forms are defined on J^rY we can write η in the form $\eta = d\,\Psi^{k+1}_r(\eta) + \big(\eta - d\,\Psi^{k+1}_r(\eta)\big)$ as an element of $d\,\Omega^k_{r(c)} + \Omega^{k+1}_{r(c)} = \Theta^{k+1}_r$. Clearly, it is sufficient to prove that $\Psi^{k+1}_r(\eta) \in \Omega^k_r$. Let $p_k\eta = \sum_{I \in \mathrm{multi}_k(\mathbb{Z})} A^I_\Sigma \, dt \wedge \omega^\Sigma_I$, where $A^I_\Sigma = 0$ for $I \notin \mathrm{multi}_k\,(0, \ldots, r)$. Let us denote $H^I_\Sigma = {}^\eta G^I_\Sigma$ for $I \in \mathrm{multi}_k\,(0, \ldots, r)$, $H^I_\Sigma = 0$ for $I \notin \mathrm{multi}_k\,(0, \ldots, r)$. Then $\Psi^{k+1}_r(\eta) = \sum_{I \in \mathrm{multi}_k(0,\ldots,r)} H^I_\Sigma \, \omega^\Sigma_I$. In order to prove that $\Psi^{k+1}_r(\eta) \in \Omega^k_r$ we have to prove that all the functions H^I_Σ are defined on J^rY and that $H^I_\Sigma = 0$ whenever I contains at least one r. The second condition follows immediately from the coordinate representation of the equation $\Phi^{k+1}_r(\eta) = 0$. (See Lemma 4.) Let us prove the first one. In the coordinates the equation $\Phi^{k+1}_r(\eta) = 0$ has the form

$$\sum_{I \in \mathrm{multi}_k(\mathbb{Z})} A^I_\Sigma \, dt \wedge \omega^\Sigma_I = \sum_{I \in \mathrm{multi}_k(\mathbb{Z})} \Big(d_1 H^I_\Sigma + \sum_{|\mathfrak{J}|=1} H^{I-\mathfrak{J}}_\Sigma \Big) \, dt \wedge \omega^\Sigma_I. \tag{7}$$

Both of the systems

$$\big(A^I_\Sigma\big)_{\substack{I\\ \Sigma}}, \; \Big(d_1 H^I_\Sigma + \sum_{|\mathfrak{J}|=1} H^{I-\mathfrak{J}}_\Sigma \Big)_{\substack{I\\ \Sigma}}$$

are antisymmetric and for each $I \notin \mathrm{multi}_k(0, \ldots, r)$ we have $A^I_\Sigma = 0$, $d_1 H^I_\Sigma + \sum_{|\mathfrak{J}|=1} H^{I-\mathfrak{J}}_\Sigma = 0$ so that the equation (7) is equivalent to the following system of equations

$$\begin{aligned} & d_1 H^I_\Sigma + \sum_{|\mathfrak{I}|=1} H^{I-\mathfrak{I}}_\Sigma = A^I_\Sigma \\ & I \in \operatorname{multi}_k(0,\dots,r),\ \Sigma \in \operatorname{multi}_k(1,\dots,m). \end{aligned} \tag{8}$$

Let us prove by the mathematical induction that all the functions H^I_Σ are defined on J^rY. For $|I| = 0$, $\Sigma \in \operatorname{multi}_k(1,\dots,m)$ we get from the system (8) the equation $d_1 H^{0,\dots,0}_\Sigma = A^{0,\dots,0}_\Sigma$ which shows that $H^{0,\dots,0}_\Sigma \in \Omega^0_r$ because $A^{0,\dots,0}_\Sigma \in \Omega^0_{r+1}$. Let us suppose that for all $I \in \operatorname{multi}_k(0,\dots,r)$ such that $|I| \le p$ and for all $\Sigma \in \operatorname{multi}_k(1,\dots,m)$ the functions H^I_Σ belong to Ω^0_r . Let $I \in \operatorname{multi}_k(0,\dots,r)$ be a multiindex such that $|I| = p+1$, let $\Sigma \in \operatorname{multi}_k(1,\dots,m)$. The corresponding equation from the system (8) gives us

$$d_1 H^I_\Sigma = A^I_\Sigma - \sum_{|\mathfrak{I}|=1} H^{I-\mathfrak{I}}_\Sigma. \tag{9}$$

The function on the right side of (9) belongs to Ω^0_{r+1} which means that H^I_Σ is defined on J^rY. □

The mappings Φ allow us to represent the variational sequence in the following way

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r] & \Omega^0 \arrow[r, "E_0"] \arrow[dr, "\overline{E}_0"'] & \Omega^1/\Theta^1 \arrow[r, "E_1"] \arrow[d, "\Phi^1_r = p_0", "\cong"'] & \Omega^2/\Theta^2 \arrow[r, "E_2"] \arrow[d, "\Phi^2_r", "\cong"'] & \Omega^3/\Theta^3 \arrow[r] \arrow[d, "\Phi^3_r", "\cong"'] & \cdots \\ & & & \operatorname{Im}\Phi^1_r \arrow[r, "\overline{E}_1"'] & \operatorname{Im}\Phi^2_r \arrow[r, "\overline{E}_2"'] & \operatorname{Im}\Phi^3_r \arrow[r, "\overline{E}_3"'] & \cdots \end{tikzcd} \tag{10}$$

where $\operatorname{Im}\Phi^1_r \subseteq \Omega^1_{r+1,X}$, $\operatorname{Im}\Phi^i_r \subseteq \Omega^i_{2r+1}$ for $i > 1$. (We use the same notation for the mapping $\Phi^k_r : \Omega^k_r \to \Omega^k_s$ and for the corresponding mapping $\Omega^k_r/\operatorname{Ker}\Phi^k_r = \Omega^k_r/\Theta^k_r \longrightarrow \Omega^k_s$.) For applications of this representation it would be convenient to know the coordinate representation of the mappings from the diagram

$$\begin{tikzcd} \Omega^1 \arrow[d, "\Phi^1_r"] & \Omega^2 \arrow[d, "\Phi^2_r"] & \Omega^3 \arrow[d, "\Phi^3_r"] \\ \operatorname{Im}\Phi^1_r \arrow[r, "\overline{E}_1"] & \operatorname{Im}\Phi^1_r \arrow[r, "\overline{E}_2"] & \operatorname{Im}\Phi^1_r. \end{tikzcd}$$

Notation. For a function $L : J^rY \to \mathbb{R}$ we denote by $\varepsilon_\sigma(L)$ the Euler–Lagrange expressions

$$\varepsilon_\sigma(L) = \sum_{l=0}^{r} (-1)^l\, d_l \frac{\partial L}{\partial q^\sigma_l}.$$

For a system $(\varepsilon_\sigma)_{\sigma=1,\dots,m}$ of functions on J^rY we denote by $H^i_{\sigma_1,\sigma_2}(\varepsilon_\sigma)$ the Helmholtz–Sonin expressions

$$H^i_{\sigma_1,\sigma_2}(\varepsilon_\sigma) = \frac{\partial \varepsilon_{\sigma_1}}{\partial q_i^{\sigma_2}} - (-1)^i \frac{\partial \varepsilon_{\sigma_2}}{\partial q_i^{\sigma_1}} - \sum_{k=i+1}^{\infty} (-1)^k \binom{k}{i} d_{k-i} \frac{\partial \varepsilon_{\sigma_2}}{\partial q_k^{\sigma_1}}.$$

(For further information see [3, 4].)

Theorem 2. (1) *Let* $\eta = A\,dt + \sum_{i=0}^{r-1} B^i_\sigma\,\omega^\sigma_i + C_\sigma\,dq^\sigma_r \in \Omega^1_r$. *Then*

$$\Phi^1_r(\eta) = p_0(\eta) = \left(A + q^\tau_{r+1} C_\tau\right) dt.$$

(2) *Let* $\eta \in \Omega^2_r$, *let* $p_1(\eta) = \sum_{i=0}^r A^i_\sigma\, dt \wedge \omega^\sigma_i$. *Then*

$$\Phi^2_r(\eta) = \left(\sum_{l=0}^{r} (-1)^l d_l\, A^l_\sigma \right) dt \wedge \omega^\sigma.$$

(3) *Let* $\eta \in \Omega^3_r$, *let* $p_2(\eta) = \sum_{i_1,i_2=0}^r A^{i_1,i_2}_{\sigma_1,\sigma_2}\, dt \wedge \omega^{\sigma_1}_{i_1} \wedge \omega^{\sigma_2}_{i_2}$. *Then*

$$\begin{aligned}
\Phi^3_r(\eta) = {} & \frac{1}{2} \sum_{l=0}^{r} (-1)^l d_l \left(A^{l,0}_{\sigma_1,\sigma_2} + A^{0,l}_{\sigma_1,\sigma_2} \right) dt \wedge \omega^{\sigma_1} \wedge \omega^{\sigma_2} \\
& + \sum_{i_2=1}^{r} \left[\sum_{j=0}^{i_2} (-1)^j \sum_{l=0}^{r-j} (-1)^l \binom{l+j}{j} d_l\, A^{j+l,i_2-j}_{\sigma_1,\sigma_2} \right] dt \wedge \omega^{\sigma_1} \wedge \omega^{\sigma_2}_{i_2} \\
& + \sum_{i_1=0}^{r-1} \left[\sum_{j=0}^{r} (-1)^j \sum_{l=0}^{r-i_1-1-j} (-1)^l \binom{l+j}{j} d_l\, A^{i_1+1+j+l,r-j}_{\sigma_1,\sigma_2} \right] dt \wedge \omega^{\sigma_1}_{i_1} \\
& \wedge \omega^{\sigma_2}_{r+1}.
\end{aligned}$$

(4) *Let* $\rho = L\,dt \in \operatorname{Im}\,\Phi^1_r$. *Then*

$$\overline{E_1} = -\varepsilon_\sigma(L)\, dt \wedge \omega^\sigma.$$

(5) *Let* $\rho = \varepsilon_\sigma\, dt \wedge \omega^\sigma \in \operatorname{Im}\,\Phi^2_r$. *Then*

$$\begin{aligned}
\overline{E}_2(\rho) = {} & \tfrac{1}{2} \sum_{i=0}^{r} H^i_{\sigma_1,\sigma_2}(\varepsilon_\sigma)\, dt \wedge \omega^{\sigma_1} \wedge \omega^{\sigma_2} \\
& - \tfrac{1}{2} \left[(-1)^0 dt \wedge \omega^{\sigma_1}_0 \wedge \omega^{\sigma_2}_{r+1}, \dots, (-1)^{r-1} dt \wedge \omega^{\sigma_1}_{r-1} \wedge \omega^{\sigma_2}_{r+1} \right] \\
& \cdot P^{-1} \begin{bmatrix} H^{r+1}_{\sigma_1,\sigma_2}(\varepsilon_\sigma) \\ \vdots \\ H^{2r}_{\sigma_1,\sigma_2}(\varepsilon_\sigma) \end{bmatrix},
\end{aligned}$$

where $P = \left[\binom{j-1}{i-1} \, d_{j-i} \right]_{1 \le i,j \le r}$.

Remark. The matrix P is triangular and all the elements on its diagonal are identical mappings. That means that det P is also the identical mapping and, consequently, P^{-1} can be defined by the usual formula.

Proof. (1),(2),(3) can be proved by putting $k = 0, 1, 2$ into Definition 1. In order to prove (4),(5) we have to express the coordinate representation of $\Phi_r^2 \circ d$, $\Phi_r^3 \circ d$ in the required form which is trivial for $\Phi_r^2 \circ d$ and a long and tedious computation for $\Phi_r^3 \circ d$. □

Remark. The expression in (4) shows that $\overline{E_1}$ is the Euler–Lagrange mapping. The assertion (5) shows that the representation of E_2 is not the same as in [2] or [4]. On the other hand, the equation $\overline{E_2}\,\Phi_r^2(\eta) = 0$ is equvalent to the Helmholtz–Sonin equations from [2] and [4] because the operator defined by the matrix P is bijective.

Acknowledgements

The author would like to thank Friedrich Hirzebruch for his invitation to the Max Planck Institut für Mathematik, Bonn, where part of this research was done. The author also wishes to thank Demeter Krupka for many helpful discussions and suggestions.

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