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SOME ASPECTS OF VARIATIONAL SEQUENCES IN MECHANICS

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ABSTRACT. We discuss intrinsic aspects of Krupka's approach to finite–order variational sequences. We recover in an intrinsic way the second–order variational calculus for affine Lagrangians by means of a natural generalisation of first–order theories. Moreover, we find an intrinsic expression for the Helmholtz morphism using a technique introduced by Kolář that we have adapted to our context.

Introduction

The theory of variational bicomplexes was established at the end of the seventies by several authors [AnDu80], [OlSh78], [Tak79], [Tul77], [Vin77]. In these works there is the idea that one can give a geometric formulation of the calculus of variations without using integrals. But, except of [AnDu80], in these works the variational bicomplex is built over the space of infinite jets of a fibred manifold. This procedure is suggested by the relatively simple structure of such spaces.

In this paper, we start from Krupka's setting of variational sequences on finite–order jet spaces [Kru90]. A finite–order bicomplex is produced when one quotients the de Rham sequence on a finite–order jet space by means of an intrinsically defined subsequence. This finite–order approach has been fruitfully applied to a concrete relativistic theory in [MoVi95]; the idea of a study of the variational bicomplex by means of intrinsic techniques has its origin in this work.

Here, we restrict ourselves to the case of the first–order variational bicomplex on a fibred manifold whose base is one–dimensional. We give isomorphisms of the quotient sheaves of the bicomplex with subsheaves of the sheaves of forms on a jet space of suitable order. This order is always found as the minimal among all possible candidates; this aspect is not present in the infinite jet formalism. As a by–product, we find that the first–order variational bicomplex allows to deal with second–order affine Lagrangians and the related Euler–Lagrange morphism,

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rather than first–order ones. And it is worth to point out that most of second–order horizontal Lagrangians known in physics are affine. It is also seen that the sheaves that we find contain the sheaves of the standard objects of the variational calculus as proper subsheaves. Finally, we give a characterisation of the Helmholtz morphism, showing that the well–known local coordinate expression (see [GiMa90], [Kru90]) is intrinsic.

Throughout the paper, we will use as a fundamental tool the structure form on jet spaces, developed in [MaMo83]. Moreover, we make use of intrinsic techniques that are developed by means of the language of [Cos94], and which were first introduced in [Kol83].

We end the introduction with some mathematical conventions. In this paper, manifolds are connected and C^∞, and maps between manifolds are C^∞. All morphisms of fibred manifolds (and hence bundles) will be morphisms over the identity of the base manifold, unless otherwise specified. As for sheaves, we will use the definitions and the main results given in [Wel80]. In particular, we will be concerned only with sheaves of $\mathbb{R}$–vector spaces.

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1. Jet spaces

The variational sequence in mechanics is built in a natural way (i.e., by functorial techniques) starting from the following assumption.

Assumption We assume that a fibred manifold $\pi : \boldsymbol{Y} \to \boldsymbol{X}$ is given, with $\dim \boldsymbol{X} = 1$ and $\dim \boldsymbol{Y} = m + 1$.

In coordinate expressions indices $i, j, \ldots$ will run from 1 to m and will label fibre coordinates, and the index 0 will label the coordinate on $\boldsymbol{X}$. A coordinate system on $\boldsymbol{Y}$ adapted to the fibring will be denoted by (x^0, y^i). We will denote by (∂_0, ∂_i) and (d^0, d^i) the local bases of vector fields and one–forms on $\boldsymbol{Y}$ induced by an adapted chart.

The fibred manifold $\boldsymbol{Y} \to \boldsymbol{X}$ yields in a natural way the jet manifolds $J_r\boldsymbol{Y}$. A detailed account of the theory of jets can be found in [MaMo83], [Sau89]; here we just fix necessary notation. Let us set $J_0\boldsymbol{Y} := \boldsymbol{Y}$, and $r, s \in \mathbb{N}, r \geq s$. We have the fibrings $\pi^r_s : J_r\boldsymbol{Y} \to J_s\boldsymbol{Y}$ and $\pi^r : J_r\boldsymbol{Y} \to \boldsymbol{X}$.

One–dimensional multi–indexes will be denoted by underlined latin letters $\underline{p} \in \mathbb{N}, \underline{q} \in \mathbb{N}, \ldots$. We have the adapted charts $(x^0, y^i_{\underline{p}})$ on $J_r\boldsymbol{Y}$, with $\underline{p} \leq r$ (we have set $y^i_{\underline{0}} := y^i$). Accordingly, we have the local bases $(\partial_0, \partial^{\underline{p}}_i)$ and $(d^0, d^i_{\underline{p}})$ of vector fields and one–forms on $J_r\boldsymbol{Y}$. When $r = 0, 1, 2, 3$ we will write y^i, y^i_0, y^i_{00}, y^i_{000}.

We have the natural complementary fibred morphisms (see [MaMo83])

$$Д_r : J_r\boldsymbol{Y} \to T^*\boldsymbol{X} \underset{J_{r-1}\boldsymbol{Y}}{\otimes} TJ_{r-1}\boldsymbol{Y} ,$$

$$\vartheta_r : J_r\boldsymbol{Y} \to T^*J_{r-1}\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\otimes} VJ_{r-1}\boldsymbol{Y} ;$$

whose coordinate expressions are

$$\mathit{\Pi}_r = d^0 \otimes \mathit{\Pi}_{r\,0} = d^0 \otimes (\partial_0 + y^j_{\underline{p}+1} \partial^{\underline{p}}_j) \,, \qquad 0 \leq |\underline{p}| \leq r-1 \,,$$

$$\vartheta_r = \vartheta^j_{\underline{p}} \otimes \partial^{\underline{p}}_j = (d^j_{\underline{p}} - y^j_{\underline{p}+1} d^0) \otimes \partial^{\underline{p}}_j \,, \qquad 0 \leq |\underline{p}| \leq r-1 \,.$$

The fibred morphism $\vartheta^*_r : J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} V^*J_{r-1}\boldsymbol{Y} \to J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^*J_{r-1}\boldsymbol{Y}$ yields the splitting

$$J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^*J_{r-1}\boldsymbol{Y} = \left(J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^*\boldsymbol{X} \right) \oplus \operatorname{im}\vartheta^*_r \,. \tag{1}$$

Let us denote by $\overset{k}{\Lambda}_r$ the sheaf of k–forms on $J_r\boldsymbol{Y}$. The fibring π and the form ϑ_s yield natural subsheaves of $\overset{k}{\Lambda}_r$. Let $k>0$, and $s \leq r$. We denote by $\overset{k}{\mathcal{C}}_{(r,s)}$ the sheaf of fibred morphisms $J_r\boldsymbol{Y} \to \overset{k}{\bigwedge}\operatorname{im}\vartheta^*_s$ over $J_r\boldsymbol{Y} \to J_s\boldsymbol{Y}$, which are interpreted as k–forms on $J_r\boldsymbol{Y}$. We set also $\overset{k}{\mathcal{C}}_r := \overset{k}{\mathcal{C}}_{(r,r)}$ and $\overset{0}{\mathcal{C}}_{(r,s)} := \overset{0}{\Lambda}_r$. We denote by $\overset{1}{\mathcal{H}}_r$ the sheaf of fibred morphisms $J_r\boldsymbol{Y} \to T^*\boldsymbol{X}$ over $J_{r-1}\boldsymbol{Y} \to \boldsymbol{X}$, which are interpreted as 1–forms on $J_r\boldsymbol{Y}$.

We denote by $\overset{1}{\mathcal{C}}{}^A_{(r,s)} \subset \overset{1}{\mathcal{C}}_{(r,s)}$ the sheaf of fibred morphisms $\alpha : J_r\boldsymbol{Y} \to \operatorname{im}\vartheta^*_s$ over π^r_s which factorise as $\alpha = \vartheta^*_s \circ \tilde{\alpha}$, where $\tilde{\alpha} : J_r\boldsymbol{Y} \to J_s\boldsymbol{Y} \underset{J_{s-1}\boldsymbol{Y}}{\times} V^*J_{s-1}\boldsymbol{Y}$ is an affine fibred morphism over π^r_s. In a similar way, we can define the subsheaf $\overset{k}{\mathcal{C}}{}^A_{(r,s)} \subset \overset{k}{\mathcal{C}}_{(r,s)}$. Also, we denote by $\overset{1}{\mathcal{H}}{}^A_r \subset \overset{1}{\mathcal{H}}_r$ the sheaf of fibred morphisms $\alpha : J_r\boldsymbol{Y} \to T^*\boldsymbol{X}$ that are affine fibred morphisms over $\pi^{r-1} : J_{r-1}\boldsymbol{Y} \to \boldsymbol{X}$. The subsheaves with the superscript A are characterised by the fact that the components of their coordinate expressions are affine functions.

The splitting (1) provides the sheaf morphisms

$$h : \overset{k}{\Lambda}_{r-1} \to \overset{k-1}{\mathcal{C}}{}^A_r \wedge \overset{1}{\mathcal{H}}_{r-1} \qquad\qquad v : \overset{k}{\Lambda}_{r-1} \to \overset{k}{\mathcal{C}}{}^A_r$$

where $h + v = {\pi^r_{r-1}}^*$, and $\overset{k-1}{\mathcal{C}}{}^A_r \wedge \overset{1}{\mathcal{H}}_{r-1}$ is done by means of the natural inclusion $\overset{1}{\mathcal{H}}_{r-1} \subset \overset{1}{\mathcal{H}}_r$. We have $h = i^{k-1}_v i_h$ and $v = 1/k!\; i^k_v$; moreover, h is surjective.

The morphisms $\mathit{\Pi}_r$ and ϑ_r induce two derivations along π^r_{r-1} of degree 1 (see [Sau89], [Cos94]), the *horizontal* and *vertical differential* d_h and d_v

$$d_h := i_h \circ d - d \circ i_h \;:\; \Lambda_{r-1} \to \Lambda_r \,,$$

$$d_v := i_v \circ d - d \circ i_v \;:\; \Lambda_{r-1} \to \Lambda_r \,,$$

where i_h and i_v are defined to be, respectively, contractions with $\mathit{\Pi}_r$ and ϑ_r. It can be proved (see [Sau89]) that d_h and d_v fulfill the property $d_h + d_v = (\pi^r_{r-1})^* \circ d$. One can find coordinate expressions of d_h and d_v in [Cos94], [Sau89]. Also, one can easily check that

$$d_h \left(\overset{k}{\mathcal{C}}_{r-1} \right) \subset \overset{k}{\mathcal{C}}_r \wedge \overset{1}{\mathcal{H}}_r \,, \qquad\qquad d_v \left(\overset{k}{\mathcal{C}}_{r-1} \right) \subset \overset{k+1}{\mathcal{C}}_r \,.$$

2. The first–order variational sequence

We consider a short form of the first–order variational sequence of Krupka [Kru90]. Let us set $\overset{0}{\Theta}_r := \{0\}$, $\overset{1}{\Theta}_r := \overset{1}{\mathcal{C}}_r$, $\overset{k}{\Theta}_r := \overset{k}{\mathcal{C}}_r + d\,\overset{k-1}{\mathcal{C}}_r$ for all $k \geq 2$. The following diagram is an exact bicomplex [Kru90]

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \overset{1}{\Theta}_1 \arrow[r, "d"] \arrow[d] & \overset{2}{\Theta}_1 \arrow[r, "d"] \arrow[d] & d\overset{2}{\Theta}_1 \arrow[r, "d"] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d] & \overset{0}{\Lambda}_1 \arrow[r, "d"] \arrow[d] & \overset{1}{\Lambda}_1 \arrow[r, "d"] \arrow[d] & \overset{2}{\Lambda}_1 \arrow[r, "d"] \arrow[d] & d\overset{2}{\Lambda}_1 \arrow[r, "d"] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d] & \overset{0}{\Lambda}_1 \arrow[r, "E_0"] \arrow[d] & \overset{1}{\Lambda}_1/\overset{1}{\Theta}_1 \arrow[r, "E_1"] \arrow[d] & \overset{2}{\Lambda}_1/\overset{2}{\Theta}_1 \arrow[r, "E_2"] \arrow[d] & E_2\left(\overset{2}{\Lambda}_1/\overset{2}{\Theta}_1\right) \arrow[r, "E_3"] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 & 0 &
\end{tikzcd}
$$

We say the bottom row of the above bicomplex to be the *short variational sequence*. Our main task is to find sheaves of fibred morphisms which are isomorphic to the quotient sheaves of the short variational sequence. It is possible to introduce a first simplification of the quotient sheaves; namely, the projection h induces the sheaf isomorphism

$$\left(\overset{k}{\Lambda}_r/\overset{k}{\Theta}_r\right) \to \left(\overset{k-1}{\mathcal{C}}{}^{A}_{r+1} \wedge \overset{1}{\mathcal{H}}_r\right)/d_h\overset{k-1}{\mathcal{C}}_r \;:\; [\alpha] \mapsto [h(\alpha)]\ .$$

We notice that the natural inclusion $\overset{k}{\Lambda}_1/\overset{k}{\Theta}_1 \subset \overset{k}{\Lambda}_{r-1}/\overset{k}{\Theta}_{r-1}$ induces the natural inclusion

$$\left(\overset{k-1}{\mathcal{C}}{}^{A}_{2} \wedge \overset{1}{\mathcal{H}}_1\right)/d_h\overset{k-1}{\mathcal{C}}_1 \to \left(\overset{1}{\mathcal{C}}{}^{A}_{r} \wedge \overset{1}{\mathcal{H}}_{r-1}\right)/d_h\overset{k-1}{\mathcal{C}}_{r-1}\ , \qquad r \geq 3 \tag{2}$$

Proposition 2.1. *We have the natural sheaf isomorphism*

$$I_1 \;:\; \left(\overset{1}{\Lambda}_1/\overset{1}{\Theta}_1\right) \to \overset{1}{\mathcal{H}}{}^{A}_{2} \;:\; [\alpha] \mapsto h(\alpha)\ .$$

If $\alpha \in \overset{1}{\Lambda}_1$ has the coordinate expression $\alpha = \alpha_0 d^0 + \alpha_i d^i + \alpha^0_i d^i_0$, then we have the coordinate expression

$$h(\alpha) = (\alpha_0 + \alpha_i y^i_0 + \alpha^0_i y^i_{00})d^0\ .$$

Let us set $\overset{1}{\mathcal{V}}_1 := \overset{1}{\mathcal{H}}{}^{A}_{2}$. We say a section $L \in \overset{1}{\mathcal{V}}_1$ to be a *first–order generalised Lagrangian.* It is worth to note that the sheaf of the first–order Lagrangians of the standard literature is $\overset{1}{\mathcal{H}}_1$, and that $\overset{1}{\mathcal{H}}_1 \subset \overset{1}{\mathcal{H}}{}^{A}_{2}$.

The following Lemma generalises the standard momentum (or Legendre transform) of a Lagrangian (see [Cos94]).

Lemma 2.1. *There exists a natural surjective morphism*

$$p : \overset{1}{\mathcal{C}}_2 \wedge \overset{1}{\mathcal{H}}_2 \to \overset{1}{\mathcal{C}}_{(2,1)} \,.$$

If, in coordinates, $\alpha = \alpha_i \vartheta^i \wedge d^0 + \alpha_i^0 \vartheta_0^i \wedge d^0$ *(with* $\alpha_i, \alpha_i^0 \in \overset{0}{\Lambda}_2$*), then* $p(\alpha) = \alpha_i^0 \vartheta^i$.

Theorem 2.1. *We have the injective sheaf morphism*

$$I_2 : \left(\overset{1}{\mathcal{C}}{}_2^A \wedge \overset{1}{\mathcal{H}}_1\right) / d_h \overset{1}{\mathcal{C}}_1 \to \overset{2}{\Lambda}_3 : [\alpha] \mapsto \pi_2^{3*} \alpha + d_h p(\alpha) \,,$$

whose image is the sheaf

$$\overset{2}{\mathcal{V}}_1 := \left(\left(\overset{1}{\mathcal{C}}{}_2^A \wedge \overset{1}{\mathcal{H}}_1\right) + d_h \overset{1}{\mathcal{C}}_2\right) \cap \left(\overset{1}{\mathcal{C}}_{(3,1)} \wedge \overset{1}{\mathcal{H}}_3\right) \,.$$

Proof. One can test in a simple way that the morphism is well–defined via the property $p(d_h q) = -q$ fulfilled for any $q \in \overset{1}{\mathcal{C}}_1$. The injectivity follows from (2). Let $\alpha \in \overset{1}{\mathcal{C}}_2 \wedge \overset{1}{\mathcal{H}}_2$. Then one can prove that the form $p(\alpha)$ is the unique section of the sheaf $\overset{1}{\mathcal{C}}_2$ such that $\alpha + d_h p(\alpha) \in \left(\overset{1}{\mathcal{C}}_{(3,1)} \wedge \overset{1}{\mathcal{H}}_3\right)$ (see [Kol83] for the proof of a similar statement). This yields the desired characterisation of the image of I_2.

Let $\alpha \in \overset{1}{\mathcal{V}}_1$, and let the coordinate expression of α be $\alpha = (\alpha_0 + \alpha_i y_{00}^i) d^0$. Then, the identity $E_1((\alpha_0 + \alpha_i y_{00}^i) d^0) = I_2([h \circ d\beta])$, where $\beta = \alpha_0 d^0 + \alpha_i d_0^i$, yields $E_1(\alpha) = E_i \vartheta^i \wedge d^0$, where

$$E_i = \partial_i(\alpha_0 + y_{00}^j \alpha_j) + \mathit{Д}_{30}(\partial_0 \alpha_i + y_0^j \partial_j \alpha_i + y_{00}^j \partial_j^0 \alpha_i - \partial_i^0 \alpha_0 - y_{00}^j \partial_i^0 \alpha_j)$$

We say E_1 to be the *first–order generalised Euler–Lagrange operator*. Also, we say a section $E \in \overset{2}{\mathcal{V}}_1$ to be a *first–order generalised Euler–Lagrange morphism.*

Remark 2.1. It is of fundamental importance to note that the standard higher–order Euler–Lagrange operator (see [Kol83], [Sau89]), when applied to a second–order Lagrangian $\alpha \in \overset{1}{\mathcal{H}}{}_2^A$, produces exactly the same result as $E_1(\alpha)$.

Also, it is easy to verify that, when restricting E_1 to the sheaf $\overset{1}{\mathcal{H}}_1$ of the Lagrangians of the standard literature, we obtain the standard Euler–Lagrange operator.

Now, we devote ourselves to a description of the sheaf $E_2\left(\overset{2}{\mathcal{V}}_1\right) \simeq \left(d\overset{2}{\Lambda}_1 / d\overset{2}{\Theta}_1\right)$. Using (2) we find the natural injection

$$\left(d\overset{2}{\Lambda}_1 / d\overset{2}{\Theta}_1\right) \to \left(\overset{2}{\mathcal{C}}{}_4^A \wedge \overset{1}{\mathcal{H}}_3\right) / d_h \overset{2}{\mathcal{C}}_3 : [d\alpha] \mapsto [dE_1(\alpha)] \,.$$

We recall that to each $\alpha \in \overset{1}{\Lambda}_r \wedge \overset{1}{\mathcal{H}}_r$ there is a unique $E_\alpha \in \overset{1}{\mathcal{C}}_{(2r,1)} \wedge \overset{1}{\mathcal{H}}_{2r}$ and a unique $p_\alpha \in \overset{1}{\mathcal{C}}_{(2r-1,r)}$ such that $\pi_r^{2r*} \alpha = E_\alpha - d_h p_\alpha$ (*higher–order Euler–Lagrange morphism* [Kol83]).

From now on, in this subsection we will suppose that $\underline{p}$ is a multiindex such that $|\underline{p}| \geq 1$. We introduce a notation in order to deal with repeated derivatives of functions by means of the fields $Д_{r\,0}, Д_{r+1\,0}, \ldots$ in a coordinate open subset. Namely, if $\boldsymbol{U} \subset \boldsymbol{Y}$ is a coordinate open subset and $f \in \left(\overset{0}{\Lambda}_r\right)_{\boldsymbol{U}}$, then we set

$$J_{\underline{p}} f := Д_{r+|\underline{p}|\,0} \ldots Д_{r+1\,0} \, .$$

Let $u : \boldsymbol{Y} \to V\boldsymbol{Y}$ be a vertical vector field with coordinate expression $u = u^i \partial_i$. Then, the coordinate expression of the prolongation $u_r : J_r\boldsymbol{Y} \to VJ_r\boldsymbol{Y}$ (see [MaMo83]) is $u_r = J_{\underline{p}} u^i \partial_i^{\underline{p}}$.

The following Lemma is due to Krupka [Kru90].

Lemma 2.2. *Let $\boldsymbol{U} \subset \boldsymbol{Y}$ be a coordinate open subset, and $\beta \in \left(\overset{1}{\Lambda}_3 \wedge \overset{1}{\mathcal{C}}_{(3,1)} \wedge \overset{1}{\mathcal{H}}_3\right)_{\boldsymbol{U}}$ with coordinate expression*

$$\beta = \beta_{ij} \vartheta^i \wedge \vartheta^j \wedge d^0 + 2\beta_{ij}^{\underline{p}} \vartheta^i_{\underline{p}} \wedge \vartheta^j \wedge d^0 \, , \qquad \underline{p} = 1, 2, 3.$$

Then there exist

$$\tilde{H}_\beta[\boldsymbol{U}] \in \left(\left(\overset{1}{\mathcal{C}}_{(6,4)} \wedge \overset{1}{\mathcal{C}}_{(6,1)}\right)^A \wedge \overset{1}{\mathcal{H}}_5\right)_{\boldsymbol{U}} \, , \qquad q_\beta[\boldsymbol{U}] \in \left(\overset{1}{\mathcal{C}}_{(5,3)} \wedge \overset{1}{\mathcal{C}}_{(5,2)}\right)^A_{\boldsymbol{U}} \, ,$$

such that

$$\pi_3^{6*} \beta = \tilde{H}_\beta[\boldsymbol{U}] - d_h q_\beta[\boldsymbol{U}]$$

Proof. We proceed by splitting the coefficients of $\pi_3^{6*}\beta$ into symmetric and antisymmetric parts, as in [Kru90]. We obtain

$$\begin{aligned}
\tilde{H}_\beta[\boldsymbol{U}] &= (\beta_{ij} - J_0\beta_{ij}^0 + J_{00}\beta_{ij}^{00} - J_{000}\beta_{ij}^{000})\vartheta^i \wedge \vartheta^j \wedge d^0 \\
&\quad + (\beta_{ij}^0 + \beta_{ji}^0 - 2J_0\beta_{ji}^{00} + 3J_{00}\beta_{ji}^{000})\vartheta_0^i \wedge \vartheta^j \wedge d^0 \\
&\quad + (\beta_{ij}^{00} - \beta_{ji}^{00} + 3J_0\beta_{ji}^{000})\vartheta_{00}^i \wedge \vartheta^j \wedge d^0 + (\beta_{ij}^{000} + \beta_{ji}^{000})\vartheta_{000}^i \wedge \vartheta^j \wedge d^0 \\
q_\beta[\boldsymbol{U}] &= (-\beta_{ij}^0 + J_0\beta_{ij}^{00} - J_{00}\beta_{ij}^{000})\vartheta^i \wedge \vartheta^j \\
&\quad - (\beta_{ij}^{00} + \beta_{ji}^{00} - J_0\beta_{ij}^{000} - 2J_0\beta_{ji}^{000})\vartheta_0^i \wedge \vartheta^j \\
&\quad + \beta_{ij}^{000}\vartheta_0^i \wedge \vartheta_0^j - (\beta_{ij}^{000} - \beta_{ji}^{000})\vartheta_{00}^i \wedge \vartheta^j \, .
\end{aligned}$$

Theorem 2.2. *For every $\beta \in \overset{1}{\Lambda}_3 \wedge \overset{1}{\mathcal{C}}_{(3,1)} \wedge \overset{1}{\mathcal{H}}_3$ there is a unique*

$$H_\beta \in \left(\overset{1}{\mathcal{C}}_{(6,4)} \wedge \overset{1}{\mathcal{C}}_{(6,1)}\right)^A \wedge \overset{1}{\mathcal{H}}_5$$

such that

(1) *with respect to (2) we have $[\beta] = [H_\beta]$;*
(2) $\forall \quad u : \boldsymbol{Y} \to V\boldsymbol{Y} \, , \quad E_{i_{u_3}\beta} = 0 \qquad \Leftrightarrow \qquad H_\beta = 0.$

PROOF. On each coordinate domain $\boldsymbol{U}$, the form $\tilde{H}_\beta[\boldsymbol{U}]$ fulfills the second requirement. In fact, due to the arbitrariness of u, the condition $E_{i_{u_3}\beta} = 0$ gives rise to a set of local conditions whose symmetric and antisymmetric parts are precisely the same local conditions that arise from $\tilde{H}_\beta[\boldsymbol{U}] = 0$ (see [GiMa90] for a local expression of $E_{i_{u_3}\beta}$).

Let $\boldsymbol{U}$ and $\boldsymbol{U}'$ be two coordinate domains with non–empty intersection. Then, being $E_{i_{u_3}\beta} = 0$ an intrinsic condition, we obtain

$$\tilde{H}_\beta[\boldsymbol{U}]|_{\boldsymbol{U}\cap\boldsymbol{U}'} = 0 \qquad \Leftrightarrow \qquad \tilde{H}_\beta[\boldsymbol{U}']|_{\boldsymbol{U}\cap\boldsymbol{U}'} = 0 \ .$$

But the above formula is clearly equivalent to $\tilde{H}_\beta[\boldsymbol{U}]|_{\boldsymbol{U}\cap\boldsymbol{U}'} = \tilde{H}_\beta[\boldsymbol{U}']|_{\boldsymbol{U}\cap\boldsymbol{U}'}$. The result comes by setting $H_\beta|_{\boldsymbol{U}} := \tilde{H}_\beta[\boldsymbol{U}]$ on each coordinate domain $\boldsymbol{U} \subset \boldsymbol{Y}$.

As a consequence, we obtain an intrinsic 3–form $d_h q_\beta$. Hence, due to the fact that $\dim \boldsymbol{X} = 1$, we obtain an intrinsic 2–form q_β (see [Kol83]). Moreover, the following corollary holds.

Corollary 2.1. $E_2 \left(\overset{2}{\mathcal{V}}_1\right)$ *is isomorphic to the image of the injective morphism*

$$I_3 \ : \ \left(d\overset{2}{\Lambda}_1/d\overset{2}{\Theta}_1\right) \to \left(\overset{1}{\mathcal{C}}_5 \wedge \overset{1}{\mathcal{C}}_{(5,1)}\right)^A \wedge \overset{1}{\mathcal{H}}_4 \ : \ [d\alpha] \mapsto H_{d(E_1([\alpha]))} \ .$$

We can express E_2 by the above morphism: if $E \in \overset{2}{\mathcal{V}}_1$, then we have $E_2(E) = H_{dE}$. So, it is easy to find the coordinate expression of E_2. We say E_2 to be the *generalised Helmholtz operator*.

We can summarise the results of this section in the following theorem.

Theorem 2.3. *The first–order short variational sequence is isomorphic to the exact sequence*

$$0 \longrightarrow I\!R \longrightarrow \overset{0}{\Lambda}_1 \xrightarrow{E_0} \overset{1}{\mathcal{V}}_1 \xrightarrow{E_1} \overset{2}{\mathcal{V}}_1 \xrightarrow{E_2} E_2\left(\overset{2}{\mathcal{V}}_1\right) \xrightarrow{E_3} 0 \ .$$

Corollary 2.2. *(Global inverse problem; see* [Kru90]*). Let $E \in \overset{2}{\mathcal{V}}_1$ be a global section such that $E_2(E) = 0$. Suppose, moreover, that*

$$H^2_{de\ Rham}\boldsymbol{Y} = 0 \ .$$

Then, there exists a global section $L \in \left(\overset{1}{\mathcal{V}}_1\right)_{\boldsymbol{Y}}$ *such that $E_1(L) = E$.*

Remark 2.2. We have obtained an intrinsic Helmholtz fibred morphism that is associated to each first–order generalised Euler–Lagrange morphism via the sheaf morphism E_2. The vanishing of the Helmholtz morphism is completely equivalent to the standard local Helmholtz conditions (see, for example, [GiMa90]).

As a by–product, to each first–order generalised Euler–Lagrange morphism $E \in \overset{2}{\mathcal{V}}_1$ we find a unique intrinsic two–form $q_{dE} \in \left(\overset{1}{\mathcal{C}}_{(4,3)} \wedge \overset{1}{\mathcal{C}}_{(4,2)}\right)^A \subset \overset{2}{\mathcal{C}}{}^A_4$.

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