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INFINITESIMAL SYMMETRIES OF HALL EFFECT ON RIEMANN SURFACES

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Abstract. The aim of this paper is to give a geometrical formulation of Hall effect on Riemann surfaces. In our approach magnetic Hall fields are characterized solely in terms of a given riemannian metric. Infinitesimal symmetries and their associated Noether invariants are explicitly computed in genus g using the Poincaré-Cartan form.

Introduction

There is a huge literature on the quantum aspects of Hall effect, see for example [**M**] and references therein. However, it seems that not much attention has been paid to the classical theory of Hall effect.

In this paper we put the classical Hall effect into a geometrical framework, this will lay the theory ready to perform geometric quantization. In order to do this we will use the formulation of classical mechanics on a riemannian manifold $\mathcal{Q}$ using jet bundles, and consider electromagnetic fields as connections on principal $U(1)$-bundles over the spacetime of our theory $\mathcal{E} \simeq \mathcal{T} \times \mathcal{Q}$. But here we have some peculiarities due to the fact that our lagrangians are topological, in the sense that they are not globally defined, and therefore we have to see which objects of the theory, as presented in [**G**], are global and which are not. For instance it turns out that the Poincaré-Cartan form is not globally defined but its differentials glue together to produce a global two form on the first order jet bundle.

The usual way to study Hall effect on the plane is by taking a constant magnetic field; when one goes to a Riemann surface one has to make precise the meaning of the word constant.

With this idea in mind, we start by giving a geometric definition of the concept of a magnetic Hall field on a riemannian manifold; this is achieved by using the

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Levi-Civita connection and requiring that a magnetic Hall field be covariantly constant; it turns out that on oriented bidimensional compact manifolds every magnetic Hall field can be identified with a harmonic form on $\mathcal{Q}$, but in higher dimensions this result is no longer true in general.

After this we classify magnetic Hall fields and prove that they are completely determined by the riemannian metric. On bidimensional compact connected manifolds the result is that magnetic Hall field are discrete (we use atomic units, that is $\frac{h}{2\pi} = e = c = 1$); this appears in the theory as a first classical quantization, allowing us to give a classical interpretation of integer quantum Hall effect solely in terms of the topology of the configuration space.

The interest of finding all the symmetries of our problem comes mainly from two points, the first one is in a work done by Jackiw and Manton, see [**JM**], in which they showed that a monopole in addition to the standard $SO(3) = SU(2)$ had a $SL(2, \mathbb{R})$ time conformal symmetry, so it seems natural to investigate if our systems enjoy a similar larger symmetry. The second motivation to calculate all the symmetries lies in the fact that having all of them allows us to perform the symplectic reduction and then proceed to quantize the system.

To compute all the infinitesimal symmetries of the problem we use the Poincaré-Cartan form as in [**G**] but adapted to our kind of lagrangians.

Finally it is worth mentioning that we can integrate the system of partial differential equations to be satisfied by the infinitesimal symmetries, due to the fact that our manifolds have a natural complex structure, determined by the riemannian metric and the orientation.

1. Magnetic Hall fields on bidimensional riemannian manifolds

We want to study the infinitesimal symmetries of classical Hall effect; that is, the non relativistic motion of a charged particle subject to the action of a magnetic Hall field on a connected bidimensional manifold $\mathcal{Q}$ oriented by the volume element Ω_2 associated with a riemannian metric T_2. The spacetime associated with this problem is described by a manifold $\mathcal{E}$ of dimension 1+2 provided with a pseudoriemannian metric tensor $\bar{T}_2$. An observer makes a decomposition of spacetime in the following way $\mathcal{E} \simeq \mathcal{T} \times \mathcal{Q}$, the temporal part $\mathcal{T}$ being a one dimensional manifold oriented by a 1-form ω, in a way such that $\bar{T}_2 = {\pi_1}^*(\omega \otimes \omega) - {\pi_2}^* T_2$ with π_1 and π_2 the projections of $\mathcal{T} \times \mathcal{Q}$ onto the first and second factor respectively.

Let us start by recalling some well known facts in order to fix our notation.

Definition. *An electromagnetic field for an observer is given by a 2-form $\bar{F}$ on $\mathcal{T} \times \mathcal{Q}$ which verifies the Maxwell equations $d\bar{F} = 0$ and $\bar{\delta}\bar{F} = 0$ where $\bar{\delta}$ is the codifferential with respect to the metric $\bar{T}_2$.*

In coordinates one has $\bar{F} = \omega_E \wedge dt + \Omega_B = E_1\, dq^1 \wedge dt + E_2\, dq^2 \wedge dt + B\, dq^1 \wedge dq^2$ where the forms $\omega_E = E_1\, dq^1 + E_2\, dq^2$ and $\Omega_B = B\, dq^1 \wedge dq^2$ represent, respectively, the electric and magnetic components of the electromagnetic field for this observer. If $\omega_E = 0$, that is $\bar{F} = \Omega_B = B(t, q^1, q^2) dq^1 \wedge dq^2$ the observer will say that the electromagnetic field is a magnetic field. And he will say that it is time independent if moreover $B = B(q^1, q^2)$ is time independent. Therefore time independent magnetic fields can be described as the pullback under π_2 of certain 2-forms F on $\mathcal{Q}$; that is

Definition. *A time independent magnetic field is a global 2-form $\mathcal{B} = \pi_2{}^* F$ which verifies the Maxwell equations $d\mathcal{B} = 0$ and $\bar{\delta}\mathcal{B} = 0$, F being a global 2-form on $\mathcal{Q}$. (Notice that equation $d\mathcal{B} = 0$ is automatically verified because $dF = 0$ on $\mathcal{Q}$ since $dim\, \mathcal{Q} = 2$).*

Proposition. *Let F be a 2-form on $\mathcal{Q}$. The necessary and sufficient condition for $\bar{F} = \pi_2{}^* F$ to satisfy the Maxwell equations is that F satisfies $\delta F = 0$ where δ is the codifferential on $\mathcal{Q}$ with respect to the metric tensor T_2. In that case $\bar{F} = \pi_2{}^* F$ is a time independent magnetic field.*

Corollary. *If $\mathcal{Q}$ is a connected compact bidimensional manifold without boundary, then the necessary and sufficient condition for $\bar{F} = \pi_2{}^* F$ to be a time independent magnetic field is that F be a harmonic 2-form on $\mathcal{Q}$.*

Corollary. *Every time independent magnetic field on a connected, compact bidimensional manifold $\mathcal{Q}$ without boundary, is of the form $\mathcal{B} = \lambda\, \Omega_2$, Ω_2 being the pullback under π_2 of the surface element on $\mathcal{Q}$ associated with the metric tensor T_2.*

In the theory of Hall effect on a plane, one considers constant magnetic fields, that is, time independent and spatially constant fields. The study of Hall effect on other bidimensional manifolds requires taking time independent magnetic fields (as they have been defined previously) and to generalize the concept of spatial constancy according to the riemannian geometry of each manifold. Thus we shall give the following

Definition. *An electromagnetic field $\bar{F}$ is covariantly constant if $\bar{D}^{\bar{\nabla}} \bar{F} = 0$ for every vector field $\bar{D}$ on $\mathcal{T} \times \mathcal{Q}$, $\bar{\nabla}$ being the Levi-Civita connection associated with the metric tensor $\bar{T}_2$.*

Definition. *We shall call magnetic Hall field $\mathcal{B}_{Hall}$ every covariantly constant magnetic field on $\mathcal{T} \times \mathcal{Q}$.*

Since the dimension of the manifold $\mathcal{Q}$ is two, one has

Proposition. *The following conditions are equivalent:*

(1) $\mathcal{B}_{Hall}$ is a Hall field.

(2) $\mathcal{B}_{Hall} = \pi_2{}^* F$ with $D^{\nabla} F = 0$ for every vector field D on $\mathcal{Q}$, ∇ being the Levi-Civita connection associated with the metric tensor T_2 and F a 2-form on $\mathcal{Q}$.

(3) $\mathcal{B}_{Hall}$ is a time independent magnetic field.

(4) $\mathcal{B}_{Hall} = \pi_2{}^* F$, F being a co-closed 2-form on $\mathcal{Q}$.

Remark. *In spatial dimensions greater than two, the preceding proposition is not true in general.*

In the case in which $\mathcal{Q}$ is compact and without boundary, every Hall field $\mathcal{B}_{Hall} = {\pi_2}^* F$ comes from a harmonic 2-form F on $\mathcal{Q}$, and in this sense we can identify them.

Proposition. *If $\mathcal{Q}$ is compact and without boundary, every Hall field is of the form $\mathcal{B}_{Hall} = \lambda\,\Omega_2$ with $\lambda \in \mathbb{R}$.*

In these conditions the local description of a magnetic Hall field will therefore be $\mathcal{B}_{Hall} = \lambda\,\sqrt{detG}\,dq^1 \wedge dq^2$.

Using the language of connections on principal fibre bundles [**B**], a Hall field on a configuration space $\mathcal{Q}$ can be defined by the curvature form Ω_ω of a connection ω on a principal fibre bundle $p : \mathcal{P} \longrightarrow \mathcal{Q}$ with structural group $U(1)$. Let $\{U_\alpha\}$ be an open contractible cover of $\mathcal{Q}$, and $s_\alpha : U_\alpha \longrightarrow \mathcal{P}$ local sections trivializing (locally) the principal fibre bundle. For each open set U_α, let $s_\alpha^*(\omega) = A^{(\alpha)}$ and $s_\alpha^*(\Omega_\omega) = F^{(\alpha)}$ which are interpreted as the local vector potential and the magnetic Hall field on U_α, respectively. In the intersection of two open sets $U_\alpha \cap U_\beta$, we have $A^{(\beta)} - A^{(\alpha)} = d\psi^{(\alpha\beta)}$, $F^{(\beta)} = F^{(\alpha)}$. From the preceding relations we see that on $\mathcal{Q}$ there exists a global 2-form F which, if the condition $\delta F = 0$ holds, corresponds to a magnetic Hall field. But in general, there is not a global 1-form A on $\mathcal{Q}$ such that $F = dA$, although locally it is always verified $F^{(\alpha)} = dA^{(\alpha)}$.

If the fibre bundle $\mathcal{P}$ is trivial, one can find a global section $s : \mathcal{Q} \longrightarrow \mathcal{P}$ such that $\mathcal{P} = \mathcal{Q} \times U(1)$. This allows us to define globally the vector potential A and the magnetic Hall field F on $\mathcal{Q}$ as: $s^*(\omega) = A$ and $s^*(\Omega_\omega) = F$, verifying $F = dA$. In this case, as we shall see, the motion of a charged particle of mass m in the presence of a global magnetic Hall field on the configuration space $\mathcal{Q}$ can be described by a global lagrangian density. But a different problem appears when the fibre bundle $p : \mathcal{P} \longrightarrow \mathcal{Q}$ is not trivial; in this case the lagrangian for the Hall effect has only a local character.

On the other hand, it is known [**KN**] that $U(1)$- principal fibre bundles $\mathcal{P}$ over $\mathcal{Q}$ are classified by their first Chern class $C_1(\mathcal{P}) \in H^2(\mathcal{Q}, \mathbb{Z})$.

Theorem. *Magnetic Hall fields on oriented compact connected bidimensional riemannian manifolds are discrete, that is*

$$F_{Hall} = n\,\frac{2\pi}{A(\mathcal{Q})}\,\Omega_2 \quad \forall n \in \mathbb{Z}$$

the integer $n = c_1(\mathcal{P})$ being the Chern number of the principal fibre bundle.

Proof. Bearing in mind the Chern-Weil theorem, the first Chern class $C_1(\mathcal{P})$ can be represented by $[\frac{1}{2\pi}\,F_{Hall}]$, but every magnetic Hall field is $F_{Hall} = \lambda\,\Omega_2$ then $C_1(\mathcal{P}) = [\frac{1}{2\pi}\,\lambda\,\Omega_2]$ with λ a constant. Since $\mathcal{Q}$ is compact the first Chern number is the integer

$$c_1(\mathcal{P}) = \frac{1}{2\pi}\int_{\mathcal{Q}} F_{Hall} = \frac{1}{2\pi}\int_{\mathcal{Q}} \lambda\,\Omega_2 = \frac{1}{2\pi}\lambda\int_{\mathcal{Q}} \Omega_2 = \frac{1}{2\pi}\lambda A(\mathcal{Q})$$

where $A(\mathcal{Q})$ is the area of $\mathcal{Q}$. Therefore, one has $\lambda = \frac{2\pi}{A(\mathcal{Q})}\, c_1(\mathcal{P})$; that is, every Hall field on $\mathcal{Q}$ is of the form $F_{Hall} = \frac{2\pi}{A(\mathcal{Q})}\, c_1(\mathcal{P})\, \Omega_2$ with $c_1(\mathcal{P}) \in \mathbb{Z}$. So the claim is proved.

In local coordinates

$$F_{Hall} = \frac{2\, n\pi\, \sqrt{det\, g}}{A(\mathcal{Q})}\, dq^1 \wedge dq^2 = n\, B_{Hall}\, dq^1 \wedge dq^2$$

$B_{Hall} = \frac{2\pi \sqrt{det\, g}}{A(\mathcal{Q})}$ being the fundamental charge of magnetic Hall fields on $\mathcal{Q}$.

This is our geometrical interpretation of Hall effect: In thin bidimensional plates of certain types of materials in constant magnetic fields at low temperatures, it is verified that the conductivity of the above mentioned materials is quantized, having "plateaux" as the magnetic field varies. Our interpretation of this phenomenon is that these physical conditions produce boundary conditions such that the electrons may be considered as moving in a connected compact bidimensional manifold $\mathcal{Q}$ on which a magnetic Hall field acts; the latter is quantized according to our previous description. If one varies continuously the external magnetic field the magnetic Hall field induced on $\mathcal{Q}$ can vary only in a discrete way because it is quantized. That is, the "plateaux" appear because magnetic Hall fields are quantized on surfaces by simple topological reasons. In such case, if we denote by R_H the Hall resistivity, it will be directly proportional to the magnetic Hall field and inversely to the density of carriers N which will be, in principle, a constant depending on the material:

$$R_H = \frac{c_1(P)\, B_{Hall}}{N}$$

In this way one obtains a simple geometrical interpretation of "plateaux" in the Hall effect.

2. Some local and global aspects of the variational formulation of Hall effect

The classical description of the motion of a charged particle of mass m in a magnetic Hall field $\mathcal{B}_{Hall} = {\pi_2}^* F$ on a bidimensional riemannian manifold $\mathcal{Q}$ is formulated as a variational problem [**G**] on the first order jet bundle of sections $\pi : J^1(\mathcal{T} \times \mathcal{Q}/\mathcal{T}) \longrightarrow \mathcal{T} \times \mathcal{Q}$ of the regular projection $\pi_1 \colon \mathcal{T} \times \mathcal{Q} \longrightarrow \mathcal{T}$ associated with a lagrangian density which, in general, is defined only locally. In $\bar{U}_\alpha = \pi^{-1}(I \times U_\alpha) \subset J^1(\mathcal{T} \times \mathcal{Q}/\mathcal{T})$ the lagrangian is given by

$$\mathcal{L}^{(\alpha)}_{Hall} = \frac{1}{2} m\, \dot{q}^i_{(\alpha)} g^{(\alpha)}_{ij}\, \dot{q}^j_{(\alpha)} + A^{(\alpha)}_i \dot{q}^i_{(\alpha)} \qquad with \qquad A^{(\alpha)}_i = A^{(\alpha)}_i(q^1_{(\alpha)}, q^2_{(\alpha)})$$

where $\{q^i_{(\alpha)}\}$ are local coordinates on $\{U_\alpha\}$, $T^{(\alpha)}_2 = g^{(\alpha)}_{ij}\, dq^i_{(\alpha)} \otimes dq^j_{(\alpha)}$ is the expression for the riemannian metric and $A^{(\alpha)} = A^{(\alpha)}_i dq^i_{(\alpha)}$ is a connection 1-form on $\{U_\alpha\}$ describing a vector potential for the magnetic Hall field $F^{(\alpha)} = dA^{(\alpha)} = F^{(\alpha)}_{ij}\, dq^i_{(\alpha)} \wedge dq^j_{(\alpha)}$.

It is well known that this lagrangian is, in general, defined only locally. If $\bar{U}_\alpha \cap \bar{U}_\beta \neq \emptyset$, we have

$$\mathcal{L}^{(\alpha)}_{Hall} = \mathcal{L}^{(\beta)}_{Hall} + \frac{\partial \psi^{(\alpha\beta)}}{\partial q^i_{(\alpha)}} \dot{q}^i_{(\alpha)}$$

since $A^{(\beta)} = A^{(\alpha)} + d\psi^{(\alpha\beta)} = A^{(\alpha)} + \frac{\partial \psi^{(\alpha\beta)}}{\partial q^i_{(\alpha)}}\, dq^i_{(\alpha)}$

Analogously the Poincaré-Cartan form is defined only locally, since

$$\begin{aligned}\Theta^{(\alpha)}_{Hall} &= -\frac{\partial\, \mathcal{L}^{(\alpha)}}{\partial\, \dot{q}^{(\alpha)}_j}\, dq^{(\alpha)}_j + [\frac{\partial\, \mathcal{L}^{(\alpha)}}{\partial\, \dot{q}^{(\alpha)}_j}\, \dot{q}^{(\alpha)}_j - \mathcal{L}^{(\alpha)}]\, dt = \\ &= -(m\, \dot{q}^{(\alpha)}_i g^{(\alpha)}_{ij} + A^{(\alpha)}_j)\, dq^{(\alpha)}_j + \frac{1}{2} m\, \dot{q}^{(\alpha)}_i g^{(\alpha)}_{ij}\, \dot{q}^{(\alpha)}_j\, dt\end{aligned}$$

and on the non empty intersections $\bar{U_\alpha} \cap \bar{U_\beta} \neq \emptyset$

$$\Theta^{(\alpha)}_{Hall} = \Theta^{(\beta)}_{Hall} + d\psi_{\alpha\beta}$$

However, let us observe that

$$d\Theta^{(\alpha)}_{Hall} = d\Theta^{(\beta)}_{Hall}$$

and therefore there exists a global 2-form ω_2 on $J^1(\mathcal{T} \times \mathcal{Q}/\mathcal{T})$ such that locally $\omega^{(\alpha)}_2 = d\Theta^{(\alpha)}_{Hall}$. We have that $\omega_2 = d\Theta_{kin} - F$, where Θ_{kin} is the contribution of the global kinetic term and F is (the pull-back to the 1-jet bundle) the magnetic Hall field globally defined on $\mathcal{Q}$. But in general, $\omega_2 \neq d\Theta$.

3. Infinitesimal symmetries and Noether invariants of Hall effect

When the Hall lagrangian density $\mathcal{L}_{Hall}\, dt$ is globally defined, we can define the infinitesimal automorphisms and Noether invariants of Hall effect in the usual way [**G**]:

Definition. *We shall say that a vector field $\bar{D}$ on $J^1(\mathcal{T} \times \mathcal{Q}/\mathcal{T})$ is an infinitesimal automorphism of the Hall effect described by a global lagrangian density $\mathcal{L}_{Hall}\, dt$, if $\bar{D}^L\, \Theta_{Hall} = -dg_D$, Θ_{Hall} being the associated Poincaré-Cartan form, which in this case is globally defined. Then $f_{\bar{D}} = i_{\bar{D}}\, \Theta + g_{\bar{D}}$ defines a function on the corresponding phase space which is called the Noether invariant associated with the infinitesimal automorphism $\bar{D}$.*

We shall call infinitesimal symmetry to every vector field D on $\mathcal{T} \times \mathcal{Q}$ such that the 1-jet prolongation of it $j^1(D)$ be an infinitesimal automorphism of Hall effect.

However, the non-globallity of the lagrangian and of the Poincaré-Cartan form in the Hall effect compels us to revise the local or global aspects of infinitesimal symmetries and Noether invariants of a variational problem of this type. For similar ideas see [**JM**] and [**DH**].

Definition. *We shall say that a vector field D on $\mathcal{T}\times\mathcal{Q}$ is an infinitesimal symmetry of Hall effect if the 1-jet prolongation of it $j^1(D)$ verifies that $[j^1(D)]^L\,\Theta^{(\alpha)} = -dg_D^{(\alpha)}$ on each open set.*

Remark. *It is easy to see that the set of infinitesimal symmetries $\mathcal{D}$ is a Lie algebra.*

On the intersection of two open sets, where $\Theta^{(\alpha)} = \Theta^{(\beta)} + d\psi_{\alpha\beta}$, we have $g_D^{(\alpha)} = g_D^{(\beta)} - [j^1(D)](\psi_{\alpha\beta})$. We can define a global function f_D such that on each open set

$$f_D^{(\alpha)} = i_{[j^1(D)]}\,\Theta^{(\alpha)} + g_D^{(\alpha)}$$

One easily checks that $f_D^{(\alpha)} = f_D^{(\beta)}$ on $\bar{U_\alpha}\cap\bar{U_\beta}\neq\emptyset$

Definition. *The function f_D associated with each infinitesimal symmetry D is called the Noether invariant associated with D.*

In order to calculate the infinitesimal symmetries of Hall effect we will make use of the natural complex structure of $\mathcal{Q}$ (recall that $\mathcal{Q}$ is a bidimensional oriented riemannian manifold). Let U be an open set on $\mathcal{Q}$ with coordinates $\{q^1, q^2\}$; the complex chart is $z = q^1 + iq^2$. The (local) Hall potential on U can be written now as $A_{Hall} = \mathcal{A}_z\,dz + \mathcal{A}_{\bar z}\,d\bar z$, thus the magnetic Hall field is $F_{Hall} = dA_{Hall} = F_{z\bar z}\,dz\wedge d\bar z$.

The Poincaré-Cartan form of Hall effect is

$$\Theta_{Hall} = [\lambda_z\,\dot z + \mu\,\dot{\bar z} - \mathcal{A}_z]\,dz + [\mu\,\dot z + \lambda_2\,\dot{\bar z} - \mathcal{A}_{\bar z}]\,d\bar z - \frac{1}{2}\,[\lambda_z\,\dot z^2 + \lambda_{\bar z}\,\dot{\bar z}^2 + 2\mu\,\dot z\,\dot{\bar z}]\,dt$$

where

$$\begin{aligned}\lambda_z &= -\frac{m}{4}g_{11} - \frac{2m}{4i}g_{12} + \frac{m}{4}g_{22}\\ \lambda_{\bar z} &= -\frac{m}{4}g_{11} + \frac{2m}{4i}g_{12} + \frac{m}{4}g_{22}\\ \mu &= -\frac{m}{4}g_{11} - \frac{m}{4}g_{22}\end{aligned}$$

Notice that $\lambda_z = \lambda_z(z,\bar z)$, $\lambda_{\bar z} = \lambda_{\bar z}(z,\bar z)$ and $\mu = \mu(z,\bar z)$. Moreover $\bar\lambda_z = \lambda_{\bar z}$ and $\bar\mu = \mu$. It is easy to check that $det\,g = g_{11}g_{22} - {g_{12}}^2 = \lambda_z\lambda_{\bar z} - \mu^2$.

We can write the infinitesimal symmetries in the form

$$D = f(t)\,\frac{\partial}{\partial t} + h^z(t,z,\bar z)\,\frac{\partial}{\partial z} + h^{\bar z}(t,z,\bar z)\,\frac{\partial}{\partial \bar z}$$

(notice that $h^{\bar z} = \bar h^z$). The 1-jet prolongation $j^1(D)$ of this field must verify $[j^1(D)]^L\,\Theta_{Hall} = -dg_D$ and therefore

$$d[j^1(D)]^L\,\Theta_{Hall} = 0$$

After some easy manipulations, one has the following equations for the infinitesimal symmetries

$$
\begin{aligned}
&(Ec.1)\quad \frac{\partial^2}{\partial t^2}(\lambda_z\, h^z + \mu\, h^{\bar z}) + F_{z\bar z}\,\frac{\partial h^{\bar z}}{\partial t} = 0\\
&(Ec.2)\quad \frac{\partial^2}{\partial t^2}(\mu\, h^z + \lambda_{\bar z}\, h^{\bar z}) - F_{z\bar z}\,\frac{\partial h^{z}}{\partial t} = 0\\
&(Ec.3)\quad \frac{\partial^2}{\partial \bar z\partial t}(\lambda_z\, h^z + \mu\, h^{\bar z}) + \frac{\partial(F_{z\bar z}h^{\bar z})}{\partial \bar z} = \frac{\partial^2}{\partial z\partial t}(\mu\, h^z + \lambda_{\bar z}\, h^{\bar z}) - \frac{\partial(F_{z\bar z}h^{z})}{\partial z}\\
&(Ec.4)\quad h^z\,\frac{\partial \lambda_z}{\partial z} + h^{\bar z}\,\frac{\partial \lambda_z}{\partial \bar z} + 2\lambda_z\,\frac{\partial h^z}{\partial z} + 2\mu\,\frac{\partial h^{\bar z}}{\partial z} = \lambda_z\,\frac{\partial f}{\partial t}\\
&(Ec.5 = Ec.6)\quad \lambda_z\,\frac{\partial h^z}{\partial \bar z} + \lambda_{\bar z}\,\frac{\partial h^{\bar z}}{\partial z} + h^z\,\frac{\partial \mu}{\partial z} + h^{\bar z}\,\frac{\partial \mu}{\partial \bar z} + \mu\,\frac{\partial h^z}{\partial z} + \mu\,\frac{\partial h^{\bar z}}{\partial \bar z} = \mu\,\frac{\partial f}{\partial t}\\
&(Ec.7)\quad h^z\,\frac{\partial \lambda_{\bar z}}{\partial z} + h^{\bar z}\,\frac{\partial \lambda_{\bar z}}{\partial \bar z} + 2\lambda_{\bar z}\,\frac{\partial h^{\bar z}}{\partial \bar z} + 2\mu\,\frac{\partial h^{z}}{\partial \bar z} = \lambda_{\bar z}\,\frac{\partial f}{\partial t}
\end{aligned}
$$

where $F_{z\bar z} = \frac{i}{2}\,F_{12} = \frac{i}{2}\,\lambda\,(|\lambda_z\lambda_{\bar z} - \mu^2|)]$.

It is easy to see that (Ec.1) is the complex conjugate of (Ec.2) since $h^{\bar z} = \bar h^z$, and that (Ec.4), (Ec.5 = Ec.6) and (Ec.7) are closely related to the infinitesimal symmetries of the riemannian metric tensor T_2.

There is no way to solve this system for a generic metric. But in the study of the Hall effect on Riemann surfaces one can introduce local conformal coordinates with respect to T_2 on U (that is, $g_{12} = 0$ and $g_{11} = g_{22}$) then $\lambda_z = \lambda_{\bar z} = 0$ and $\mu = -\frac{m}{2}g_{11} \neq 0$, then the preceding equations simplify to

$$
\begin{aligned}
&(Ec.1)\quad \mu\,\frac{\partial^2 h^{\bar z}}{\partial t^2} + F_{z\bar z}\,\frac{\partial h^{\bar z}}{\partial t} = 0\\
&(Ec.2)\quad \mu\,\frac{\partial^2 h^{z}}{\partial t^2} - F_{z\bar z}\,\frac{\partial h^{z}}{\partial t} = 0\\
&(Ec.3)\quad \frac{\partial^2}{\partial \bar z\partial t}(\mu\, h^{\bar z}) + \frac{\partial(F_{z\bar z}h^{\bar z})}{\partial \bar z} = \frac{\partial^2}{\partial z\partial t}(\mu\, h^z) - \frac{\partial(F_{z\bar z}h^{z})}{\partial z}\\
&(Ec.4)\quad \frac{\partial h^{\bar z}}{\partial z} = 0\\
&(Ec.5 = Ec.6)\quad h^z\,\frac{\partial \mu}{\partial z} + h^{\bar z}\,\frac{\partial \mu}{\partial \bar z} + \mu\,\frac{\partial h^z}{\partial z} + \mu\,\frac{\partial h^{\bar z}}{\partial \bar z} = \mu\,\frac{\partial f}{\partial t}\\
&(Ec.7)\quad \frac{\partial h^{z}}{\partial \bar z} = 0
\end{aligned}
$$

and these allows us to integrate the system in these cases.

4. Some examples

Example 1 : Hall effect on the euclidean plane

The Hall lagrangian on the euclidean plane $\mathbb{R}^2$ is globally defined and in cartesian coordinates is $\mathcal{L}_{Hall} = \frac{1}{2}m(\dot x^2 + \dot y^2) + \frac{eB}{2}(-y\dot x + x\,\dot y)$ where $A = \frac{B}{2}(x dy - y\,dx)$ is the vector potential associated with the Hall field $F_{Hall} = B\,dx \wedge dy$. In this case we have $\lambda_z = \lambda_{\bar z} = 0$, $\mu = -\frac{m}{2}$ and $F_{12} = B$ with B a constant.

Theorem. *The infinitesimal symmetries of the lagrangian of the Hall effect on the euclidean plane* $\mathbb{R}^2$ *form an eight-dimensional real Lie algebra generated by the following vector fields*

$$\begin{aligned}
D_1 &= cos\,\omega t\,\frac{\partial}{\partial t} - \frac{\omega}{2}[sin\,\omega t\,x - cos\,\omega t\,y]\,\frac{\partial}{\partial x} - \frac{\omega}{2}[sin\,\omega t\,y + cos\,\omega t\,x]\,\frac{\partial}{\partial y}\\
D_2 &= sin\,\omega t\,\frac{\partial}{\partial t} + \frac{\omega}{2}[cos\,\omega t\,x + sin\,\omega t\,y]\,\frac{\partial}{\partial x} + \frac{\omega}{2}[cos\,\omega t\,y - sin\,\omega t\,x]\,\frac{\partial}{\partial y}\\
D_3 &= \frac{\partial}{\partial t}\\
D_4 &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\\
D_5 &= sin\,\omega t\frac{\partial}{\partial x} + cos\,\omega t\frac{\partial}{\partial y}\\
D_6 &= cos\,\omega t\frac{\partial}{\partial x} - sin\,\omega t\frac{\partial}{\partial y}\\
D_7 &= \frac{\partial}{\partial x}\\
D_8 &= \frac{\partial}{\partial y}
\end{aligned}$$

where $\omega = \frac{B}{m}$ *is the Larmor angular velocity.*

Corollary. *The Noether invariants associated with the preceding infinitesimal symmetries are*

$$\begin{aligned}
f_1 &= -\frac{m\omega}{2}\,[(x\,\dot{x} - y\,\dot{y})\,sin\,\omega t - (x\,\dot{y} + y\,\dot{x})\,cos\,\omega t] - \frac{m}{2}\,(\dot{x}^2 + \dot{y}^2)\,cos\,\omega t\\
f_2 &= \frac{m\omega}{2}\,[(x\,\dot{x} - y\,\dot{y})\,cos\,\omega t + (x\,\dot{y} + y\,\dot{x})\,sin\,\omega t] - \frac{m}{2}\,(\dot{x}^2 + \dot{y}^2)\,sin\,\omega t\\
f_3 &= -\frac{m}{2}\,(\dot{x}^2 + \dot{y}^2)\\
f_4 &= -m\,(x\,\dot{y} - y\,\dot{x}) + \frac{m\,\omega}{2}\,(x^2 + y^2)\\
f_5 &= m\,(\dot{x}\,sin\,\omega t + \dot{y}\,cos\,\omega t)\\
f_6 &= m\,(\dot{x}\,cos\,\omega t - \dot{y}\,sin\,\omega t)\\
f_7 &= m\,\dot{x} + m\,\omega\,y\\
f_8 &= m\,\dot{x} - m\,\omega\,y
\end{aligned}$$

Example 2 : Hall effect on the bidimensional sphere

There is not a global Hall lagrangian on the sphere; that is, the lagrangian of a charged particle of mass m in a magnetic Hall field $F_{Hall} = \frac{k}{2R^2}\,\Omega_2$ on the sphere of radius R can be defined only locally. Using stereographic coordinates we can write it on an open set as

$$\mathcal{L}_{Hall} = \frac{2mR^4}{(R^2 + z\bar{z})^2}\,\dot{z}\dot{\bar{z}} - \frac{ik}{2(R^2 + z\bar{z})}\,(\bar{z}\dot{z} - z\dot{\bar{z}})$$

Theorem. *The infinitesimal symmetries of the lagrangian of the Hall effect on the sphere form a four-dimensional real Lie algebra generated by the vector fields*

$$\begin{aligned}
D_1 &= \frac{\partial}{\partial t} \\
D_2 &= (1+\frac{z^2}{R^2})\,\frac{\partial}{\partial z} + (1+\frac{\bar{z}^2}{R^2})\,\frac{\partial}{\partial \bar{z}} \\
D_3 &= i(1-\frac{z^2}{R^2})\,\frac{\partial}{\partial z} + i(\frac{\bar{z}^2}{R^2}-1)\,\frac{\partial}{\partial \bar{z}} \\
D_4 &= i(z\,\frac{\partial}{\partial z} - \bar{z}\,\frac{\partial}{\partial \bar{z}})
\end{aligned}$$

From this we see that the vector fields $< D_2, D_3, D_4 >$ generate the Lie algebra of $SO(3) = SU(2)/\mathbb{Z}_2$.

Corollary. *The Noether invariants associated with the preceding infinitesimal symmetries are*

$$\begin{aligned}
f_{D_1} &= \frac{2mR^4}{(R^2+z\bar{z})^2}\,\dot{z}\dot{\bar{z}} \\
f_{D_2} &= \frac{-2mR^2}{(R^2+z\bar{z})^2}\,(z^2\dot{\bar{z}}+\bar{z}^2\dot{z}) - \frac{2mR^4}{(R^2+z\bar{z})^2}\,(\dot{z}+\dot{\bar{z}}) - \frac{ik}{R^2+z\bar{z}}\,(z-\bar{z}) \\
f_{D_3} &= \frac{2imR^2}{(R^2+z\bar{z})^2}\,(z^2\dot{\bar{z}}-\bar{z}^2\dot{z}) + \frac{2imR^4}{(R^2+z\bar{z})^2}\,(\dot{z}-\dot{\bar{z}}) - \frac{k}{R^2+z\bar{z}}\,(z+\bar{z}) \\
f_{D_4} &= \frac{2mR^4}{(R^2+z\bar{z})^2}\,i(\bar{z}\dot{z}-z\dot{\bar{z}}) + \frac{kR^2}{R^2+z\bar{z}} - \frac{k}{2}
\end{aligned}$$

it is easy to see that these are the energy and the angular momenta for the particle.

These ideas can be extended to arbitrary genus, the results will be published elsewhere.

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