DIFFERENTIAL GEOMETRY AND APPLICATIONS
Proc. Conf., Aug. 28 – Sept. 1, 1995, Brno, Czech Republic
Masaryk University, Brno 1996, 513–522

BRST COHOMOLOGY OF THE HALL EFFECT ON THE SPHERE

Antonio López Almorox and Carlos Tejero Prieto

Abstract. Symplectic reduction associated with the invariance under rotations of Hall effect on a sphere is analyzed. Angular momenta are in this case irreducible second class constraints and the subvariety of constant energy determines a non trivial $U(1)$-principal fibre bundle over a sphere whose radius depends on the energy and the topological charge of the magnetic Hall field. BRST differentials are constructed and their cohomologies are studied, giving a geometrical interpretation of the ghost and antighost fields. The relationship between energy and angular momenta allow us to define a super-Poisson structure for the trajectories with constant angular momentum but the induced BRST charge must be modified by a derivation of total ghost degree -1 in order to obtain the BRST-operator.

0. Introduction

One of the mathematical treatments of degenerate dynamical systems (that is, dynamical systems with symmetries or constraints) is the symplectic reduction. The differential geometric aspects of this procedure are very well known and there is a huge mathematical and physical literature about it ([1]-[3]). The cohomological aspects of this reduction have been studied in the last years by different authors with the idea of giving a quantization procedure for constrained systems ([4]-[10]). The idea is to determine the ring of functions of the reduced phase space of degenerate dynamical system as the zeroth cohomology of a differential complex determined by the symmetries or constraints of the system and the radical (or free gauges) of the submanifold of constraints.

As is well known, a magnetic Hall fields can be considered as 2-forms on a riemannian configuration space which are proportional to the area element. With this geometrical point of view and using the Poincaré-Cartan form of the lagrangian formalism [11], the authors have calculated explicitly all the infinitesimal symmetries and Noether invariants of the two dimensional systems of non-relativistic

1991 *Mathematics Subject Classification.* 17B81, 58F06, 58F05, 53C80.
Key words and phrases. BRST Cohomology, Hall effect, Symplectic reduction.
Partially supported by the C.I.C.Y.T.research project number PB-92-0308.
This paper is in final form and no version of it will be submitted for publication elsewhere.

charged particles in a magnetic Hall field [12]. In order to study the geometrical aspects of the BRST and geometric quantizations of Hall effect, here we study a particular case: the Hall effect on the two dimensional sphere but from a symplectic and BRST points of view. We analyze the symplectic reduction associated with the invariance under the $SO(3)$ transformations of this system. Angular momenta are in this case irreducible second class constraints and the subvariety of constant energy determines a non trivial $U(1)$-principal fibre bundle over a sphere whose radius depends on the energy and the topological charge of the Hall magnetic field. BRST operators for the angular momenta and energy reductions are constructed in a way closely related to the one given in [6].

This paper is organized as follows: In section 1, we study the non relativistic motion of a charged particle in a magnetic Hall field on a bidimensional sphere. The SO(3) symmetry and the explicit form of the Noether invariants are used in order to prove that the particle moves in a circle on the sphere with constant angular velocity determined by the value of the angular momenta and the topological charge (Chern class) of the Hall field. Then the symplectic reduction of this problem is completely given. Using the previous results, we construct, in section 2, a BRST differential for the Hall effect and its cohomology is studied giving a geometrical interpretation of the ghost and antighost fields. In section 3, we study the submanifold of constant energy given as the preimage of a coadjoint orbit under the moment map associated with the SO(3) symmetry and we will see that this submanifold is a non trivial principal circle bundle over another sphere which can be considered as the reduced phase space of the problem. Finally, the induced BRST charge and the super-Poisson structure are studied.

1. Symplectic reduction of the Hall effect on the sphere

The phase space associated with the bidimensional motion of a non relativistic charged particle of mass m in a magnetic Hall field on a sphere S^2 of radius R is the symplectic manifold (T^*S^2, Ω_2) where the symplectic 2-form is $\Omega_2 = \omega_2 + \pi^*(F_{Hall})$ with ω_2 the canonical symplectic form on the cotangent bundle $\pi : T^*S^2 \longrightarrow S^2$ and F_{Hall} is the 2-form on S^2 that describes the magnetic Hall field on the sphere. The Hall potential A_{Hall} (defined only locally) is interpreted as a $U(1)$-connection 1-form on the sphere and it is proved [12] that the curvature $F_{Hall} = \frac{k}{2R^2}\, w_{S^2}$ is proportional to the area element w_{S^2} of the sphere and the constant of proportionality includes the Chern class k (or topological charge of Hall field) of the $U(1)$-principal bundle where the connection is defined.

Using a lagrangian formulation, the authors [12] have proved that all the Noether invariants associated with the infinitesimal symmetries of this problem are the energy E and the angular momenta $\{J_1, J_2, J_3\}$, which in cartesian coordinates are given by $J_i = m\, \epsilon_{ijk}\, x_j \dot{x}_k - \frac{k}{2} \frac{x_i}{R}$ with $\sum_i x_i^2 = R^2$ and $\sum_i \dot{x}_i\, x_i = 0$ (see also [14] and [15] for more information about symmetries of Hall effect).

The energy E is a positive function on T^*S^2 and it fulfills the relation

$$E = \frac{1}{2mR^2}\,[J_1^2 + J_2^2 + J_3^2 - (\frac{k}{2})^2\,]$$

A hamiltonian treatment of Hall effect and its symmetries on the sphere can be done taking into account that $SO(3)$ acts in a natural way on S^2 inducing a symplectic action on (T^*S^2, Ω_2) that preserves the hamiltonian of the system and it admits an equivariant moment map $J : T^*S^2 \longrightarrow so(3)^*$with respect to the coadjoint action on $so(3)^*$. Locally it is given by $J^*(A_i) = J_i$ and $A_i = \{A_1, A_2, A_3\}$ is a fixed basis of the Lie algebra $so(3)$ with $[A_i, A_j] = \epsilon_{ijk}\,A_k$.

Let $\mu = (\mu_1, \mu_2, \mu_3)$ be a fixed point of $so(3)^*$, then $J^{-1}(\mu)$ denotes the set of points of the phase space with constant angular momenta equal to $J_i = \mu_i$, $1 \le i \le 3$. If we study trajectories with angular momentum equal to $\mu = (\mu_1, \mu_2, \mu_3)$ we must restrict our problem to the submanifold $J^{-1}(\mu)$ of T^*S^2.

Proposition [13].

1) If $\mu_1^2 + \mu_2^2 + \mu_3^2 < (\frac{k}{2})^2$ and $k \neq 0$, then $J^{-1}(\mu) = \emptyset$.

2) If $\mu_1^2 + \mu_2^2 + \mu_3^2 \ge (\frac{k}{2})^2$ and $k \neq 0$, then $J^{-1}(\mu)$ is a closed unidimensional submanifold of T^*S^2 given by the zeros of the ideal $\mathcal{I}(\mu)$ of $C^\infty(T^*S^2)$ generated by the functions $\{J_1 - \mu_1, J_2 - \mu_2, J_3 - \mu_3, \}$.

Theorem [13]. *Let $\mu_1^2 + \mu_2^2 + \mu_3^2 \ge (\frac{k}{2})^2$ be and $k \neq 0$. Let C_μ be the circle defined by the intersection of the plane $x\mu_1 + y\mu_2 + z\mu_3 + \frac{k}{2}R = 0$ with the sphere of radius R and let $\iota_\mu : C_\mu \longrightarrow S^2$ be the respective inclusion . Let us denote by $\pi_\mu : {\iota_\mu}^*[T^*S^2] \longrightarrow C_\mu$ the pull-back by ι_μ of the cotangent bundle of the sphere. Then $J^{-1}(\mu)$ is the section of π_μ given by the angular velocity around the $\vec{J}$ axis with constant module $\omega_\mu = \sqrt{\frac{\mu_1^2+\mu_2^2+\mu_3^2-(\frac{k}{2})^2}{m^2\,R^2\,s_\mu^2}}$, where s_μ is the radius of the circle C_μ.*

That is, if the angular momenta are constant and equal to μ, the particle moves with a constant angular velocity ω_μ in a circle C_μ on the sphere S^2.

From now on we will assume that $k \neq 0$ and $\mu_1^2 + \mu_2^2 + \mu_3^2 \ge (\frac{k}{2})^2$.

Let us denote by $D^{(\mu)}$ the restriction to $J^{-1}(\mu)$ of a vector field D of T^*S^2 tangent to this submanifold. Let ${\Omega_2}^{(\mu)}$ be the 2-form on $J^{-1}(\mu)$ induced by Ω_2. It is clear that $(J^{-1}(\mu), {\Omega_2}^{(\mu)})$ is not a symplectic manifold and that $rad\,{\Omega_2}^{(\mu)} = \{D^{(\mu)} \in Der\,C^\infty(J^{-1}(\mu)) \;/\; iD^{(\mu)}{\Omega_2}^{(\mu)} = \omega_0\}$ is not zero. Since the dimension of $J^{-1}(\mu)$ is one, every vector field on $J^{-1}(\mu)$ belongs to the radical of ${\Omega_2}^{(\mu)}$.

Let $\mathcal{N}(J^{-1}(\mu)) = \{f \in C^\infty(T^*S^2) \;/\; \{f, \mathcal{I}(\mu)\} \subset \mathcal{I}(\mu)]\ \}$ be the normalizer of the ideal $\mathcal{I}(\mu)$. And $\mathcal{I}_1(\mu) = \{f \in \mathcal{I}(\mu) \;/\; \{f, \mathcal{I}(\mu)\} \subset \mathcal{I}(\mu)\ \} = \mathcal{N}(J^{-1}(\mu)) \cap \mathcal{I}(\mu)$. Let us notice that $\{J_i - \mu_i, J_j - \mu_j\} = \epsilon_{ijk}\,(J_k - \mu_k) + \epsilon_{ijk}\,\mu_k$ then, if $\mu \neq 0$, the ideal $\mathcal{I}(\mu)$ is not coisotropic.

Lemma. *$f \in \mathcal{I}_1(\mu)$ if, and only if, $f = g\,[\mu_1(J_1 - \mu_1) + \mu_2(J_2 - \mu_2) + \mu_2(J_3 - \mu_3)] + [\mathcal{I}(\mu)]^2$ with g any differentiable function on T^*S^2.*

Proposition. *The radical of* ${\Omega_2}^{(\mu)}$ *is the restriction to* $J^{-1}(\mu)$ *of the hamiltonian vector fields associated with the functions of* $\mathcal{I}_1(\mu)$

Therefore $rad\,{\Omega_2}^{(\mu)}$ is generated as a $C^\infty(J^{-1}(\mu))$-module by the restriction to $J^{-1}(\mu)$ of the vector field $D_{\sum_i \mu_i(J_i-\mu_i)} = D_{\mu_1 J_1+\mu_2 J_2+\mu_3 J_3}$ because $D_{(J_i-\mu_i)} = D_{J_i}\,\forall i$. The geometric interpretation of this vector field is clear: It is the infinitesimal generator of the rotation around the axis defined by $\vec{\mu} = (\mu_1, \mu_2, \mu_3)$. In fact, the isotropy group of μ is $SO(2) = S^1$. This group acts on the circle $C_{(\mu)}$ as rotations of angular velocity $\omega_{(\mu)} = \|\vec{\omega}_{(\mu)}\|$ around $\vec{\mu}$.

In this way acting with the isotropy group $SO(2)$ on $J^{-1}(\mu) = C_{(\mu)}$, we obtain that the reduced phase space is the point $\frac{J^{-1}(\mu)}{SO(2)} = \frac{C_{(\mu)}}{SO(2)}$.

2. BRST Cohomology associated with the Hall effect on the sphere

Now let us calculate the BRST complex associated with the Hall effect ($k \neq 0$) on the sphere of radius R and for a value $\mu = (\mu_1, \mu_2, \mu_3) \in so(3)^*$, which fulfills the condition ${\mu_1}^2 + {\mu_2}^2 + {\mu_3}^2 \geq \left(\frac{k}{2}\right)^2$.

Step 1: From T^*S^2 to $J^{-1}(\mu)$: The Koszul complex

The ideal $\mathcal{I}(\mu)$ which defines the submanifold $J^{-1}(\mu)$ is a free module of rank 3 generated by the global functions $\{J_i - \mu_i\}$. The Koszul complex associated with the Hall effect is the exterior algebra of the free $C^\infty(T^*S^2)$-module generated by the dual basis $\{\rho_i\}$ of $\{J_i - \mu_i\}$ and can be identified with the exterior algebra of the free $C^\infty(T^*S^2)$-module generated by the Lie algebra of the Lie group $SO(3)$; that is

$$\mathcal{K}_{J^{-1}(\mu)} = C^\infty(T^*S^2) \otimes \Omega^{\cdot} so(3)$$

and in this case $\{\rho_i\}$ can be thought as a basis of the Lie algebra $so(3)$ such that its images by the moment map $J^* : so(3) \longrightarrow C^\infty(T^*S^2)$ are the functions $\{J_i - \mu_i\}$. The differential operator of this complex $\mathcal{K}_{J^{-1}(\mu)}$ can be defined as

$$\begin{aligned} \delta_{\mathcal{K}}(f) &= 0 \quad \forall f \in C^\infty(T^*S^2) \\ \delta_{\mathcal{K}}(\rho_i) &= J_i - \mu_i \end{aligned}$$

Since $d(J_i-\mu_i)\wedge d(J_2-\mu_2)\wedge d(J_3-\mu_3) \neq 0$, it is easy to see that μ is a regular value for the moment map $J : T^*S^2 \longrightarrow so(3)^*$ and therefore the functions $\{J_i - \mu_i\}$ are irreducible over T^*S^2 and the Koszul complex defines an acyclic resolution of $C^\infty(J^{-1}(\mu))$; that is

$$\begin{aligned} H^0(\mathcal{K}_{J^{-1}(\mu)}) &= \frac{C^\infty(X)}{\mathcal{I}(\mu)} = C^\infty(J^{-1}(\mu)) \\ H^p(\mathcal{K}_{J^{-1}(\mu)}) &= 0 \quad \forall p > 0 \end{aligned}$$

Step 2 : From $J^{-1}(\mu)$ to $\frac{J^{-1}(\mu)}{SO(2)}$: The vertical cohomology

As we have seen in the preceding section $rad\,\Omega_2{}^{(\mu)}$ is a free $C^\infty(J^{-1}(\mu))$-module of rank 1 generated by the vector field $D_{\alpha_{(\mu)}}$ which is the restriction to $J^{-1}(\mu)$ of the hamiltonian vector field $D = \frac{\omega_{(\mu)}}{\mu_1^2+\mu_2^2+\mu_3^2} D_{\mu_1 J_1+\mu_2 J_2+\mu_3 J_3}$. (Let us remember that $\frac{\omega_{(\mu)}}{\mu_1^2+\mu_2^2+\mu_3^2}[\mu_1 J_1 + \mu_2 J_2 + \mu_3 J_3] \in \mathcal{I}_1(\mu)$).

Notice that $rad\,\Omega_2{}^{(\mu)}$ can be identified with the free $C^\infty(J^{-1}(\mu))$-module generated by the Lie algebra of the isotropy group $SO(2)$ of μ; then $rad\,\Omega_2{}^{(\mu)} = C^\infty(X') \otimes so(2)$.

Let $\Omega^1(J^{-1}(\mu), rad\,\Omega_2{}^{(\mu)}) = C^\infty(X') \otimes so(2)^*$ be the dual $C^\infty(X')$-module of $rad\,\Omega_2{}^{(\mu)}$ and let $\{\eta'\}$ be the dual basis of $\{D_{\alpha_{(\mu)}}\}$, it can be thought as a basis of $so(2)^*$. The exterior algebra of the longitudinal forms for the principal fibre bundle $\pi : J^{-1}(\mu) \longrightarrow \frac{J^{-1}(\mu)}{SO(2)}$ will be denoted by $\Omega_v^{\cdot}(J^{-1}(\mu), rad\,\Omega_2{}^{(\mu)}) = C^\infty(X') \otimes \Omega^{\cdot} so(2)^*$ and the vertical differential δ_v is defined by

$$\begin{aligned} &\delta_v(f') = D_{\alpha_\mu}(f')\,\eta' \qquad \forall f' \in C^\infty(J^{-1}(\mu)) \\ &\delta_v(\eta') = 0 \end{aligned}$$

because $\eta' \wedge \eta' = 0$.

Let us denote by $H_v^{\cdot}(J^{-1}(\mu))$ the vertical cohomology . Then we have that the 0-th cohomology group is

$$\begin{aligned} H_v^0(J^{-1}(\mu)) &= \{f' \in \Omega^0(J^{-1}(\mu), rad\,\Omega_2{}^{(\mu)}) \;\;/\;\; 0 = \delta_v(f') = D_{\alpha_{(\mu)}}(f')\,\eta'\} = \\ &= C^\infty(\frac{J^{-1}(\mu)}{SO(2)}) = \mathbb{R} \end{aligned}$$

Step 3: From T^*S^2 to $\frac{J^{-1}(\mu)}{SO(2)}$: The BRST complex

According to what we have just seen, the BRST algebra of the Hall effect on the sphere for a constant value μ of the angular momentum, is $\Omega_{BRST} = \mathcal{K}_{J^{-1}(\mu)} \otimes_{C^\infty(T^*S^2)} \bar{\Omega}_v^{\cdot}(J^{-1}(\mu), rad\,\Omega_2{}^{(\mu)})$ an it can be identify as the bi-graded algebra

$$\Omega_{BRST} = C^\infty(T^*S^2) \otimes \Omega^{\cdot} so(3) \otimes \Omega^{\cdot} so(2)^*$$

Then, the three antighost $\{\rho_i\}$ are identified with a basis of the Lie algebra of the euclidean rotations, whereas the ghost $\{\eta = 1 \otimes \eta'\}$ is a basis of the Lie algebra of the isotropy group of the element μ that determines the angular momentum of the particle. Then we have that $\Omega^{p,q} = C^\infty(T^*S^2) \otimes \Omega^p so(3) \otimes \Omega^q so(2)^*$

Following [6], let us construct a BRST operator $\delta_{BRST} = \sum_0^m \delta_i$ being δ_i derivations of bi-degree $(i-1, i)$. For the non uniqueness of the BRST operator see also [4].

One has that $\delta_0 : \Omega^{p,q} \longrightarrow \Omega^{p-1,q}$ is given by

$$\begin{aligned} &\delta_0(f) = 0 \\ &\delta_0(\rho_i) = J_i - \mu_i \\ &\delta_0(\eta_j) = 0 \end{aligned}$$

Let us notice that $[\delta_0]^2 = 0$.

Now let us define a $\delta_1 : \Omega^{p,q} \longrightarrow \Omega^{p,q+1}$ in such a way that it anticommutes with δ_0 and it induces over $H^{\cdot}(\delta_0)$ the vertical differential δ_v. One has that

$$\begin{aligned}
\delta_1(f) &= \frac{\omega_{(\mu)}}{\mu_1^2+\mu_2^2+\mu_3^2}\, D_{\mu_j(J_j-\mu_j)}(f)\,\eta = \frac{\omega_{(\mu)}}{\mu_1^2+\mu_2^2+\mu_3^2}\,\{\mu_j(J_j-\mu_j), f\}\,\eta \\
\delta_1(\rho_i) &= -\frac{\omega_{(\mu)}}{\mu_1^2+\mu_2^2+\mu_3^2}\,\mu_j\epsilon_{jik}\,\rho_k\,\eta \\
\delta_1(\eta) &= 0
\end{aligned}$$

because

$$\begin{aligned}
\delta_1(J_i-\mu_i) &= \frac{\omega_{(\mu)}}{\mu_1^2+\mu_2^2+\mu_3^2}\,\{\mu_j(J_j-\mu_j), J_i-\mu_i\}\,\eta = \\
&= \frac{\omega_{(\mu)}}{\mu_1^2+\mu_2^2+\mu_3^2}\,\mu_j\epsilon_{jik}\,J_k\,\eta = \\
&= \frac{\omega_{(\mu)}}{\mu_1^2+\mu_2^2+\mu_3^2}\,[\mu_j\epsilon_{jik}\,(J_k-\mu_k) + \mu_j\epsilon_{jik}\mu_k]\,\eta = \\
&= \frac{\omega_{(\mu)}}{\mu_1^2+\mu_2^2+\mu_3^2}\,\mu_j\epsilon_{jik}\,(J_k-\mu_k)\,\eta
\end{aligned}$$

and $\mu_j\epsilon_{jik}\mu_k = 0$ for all $j = 1,2,3$. Let us notice that $\delta_0(\frac{\omega_{(\mu)}}{\mu_1^2+\mu_2^2+\mu_3^2}\,\mu_j\epsilon_{jik}\,\rho_k\,\eta) = \delta_1(J_i-\mu_i)$ from which we can calculate $\delta_1(\rho_i)$. In this case we have $[\delta_1]^2 = 0$, then

Theorem. *The Hall effect on the sphere is a finite order theory and a BRST operator $\delta_{BRST} = \delta_0 + \delta_1$ is given by*

$$\begin{aligned}
\delta_{BRST}(f) &= \frac{\omega_{(\mu)}}{\mu_1^2+\mu_2^2+\mu_3^2}\, D_{\mu_j(J_j-\mu_j)}(f)\,\eta = \frac{\omega_{(\mu)}}{\mu_1^2+\mu_2^2+\mu_3^2}\,\{\mu_j(J_j-\mu_j), f\}\,\eta \\
\delta_{BRST}(\rho_i) &= (J_i-\mu_i) - \frac{\omega_{(\mu)}}{\mu_1^2+\mu_2^2+\mu_3^2}\,\mu_j\epsilon_{jik}\,\rho_k\,\eta \\
\delta_{BRST}(\eta) &= 0
\end{aligned}$$

As it is easy to check $[\delta_{BRST}]^2 = 0$.

The BRST cohomology is given by

$$\begin{aligned}
H^n(\delta_{BRST}) &= 0 \quad \forall n < 0 \\
H^n(\delta_{BRST}) &= H_v^n(J^{-1}(\mu)) \quad \forall n \geq 0
\end{aligned}$$

and in particular, $H^0(\delta_{BRST}) = H_v^0(J^{-1}(\mu)) = \mathbb{R}$; that is, the 0-th cohomology of the BRST complex is identified with the observables of the system and it is only determined by the initial conditions of the angular momentum.

3. Super-Poisson structure on the Hall effect on the sphere

It is well known that one can choose a basis $< A_1, A_2, A_3 >$ in so(3) in a way such that the adjoint action is the natural one of $SO(3)$ on $\mathbb{R}^3$. Also, $so(3)^*$ is identified with $\mathbb{R}^3$ and the linear functions $\bar{x}_i$ defined by $\bar{x}_i(\mu) = \mu(A_i) = \mu_i$, $\forall \mu \in so(3)^*$ are a coordinate system on $so(3)^*$.

The coadjoint orbits through $\mu = (\mu_1, \mu_2, \mu_3)$ is

$$\mathcal{O}_\mu = \{(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in so(3)^* \ / \ \bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2 = \mu_1^2 + \mu_2^2 + \mu_3^2\}$$

that is, they are spheres of radius $r_{(\mu)} = [\mu_1^2 + \mu_2^2 + \mu_3^2]^{\frac{1}{2}}$. The Kirillov-Kostant symplectic form on $\mathcal{O}_\mu$, using spherical coordinates, is given by

$$\omega_2^{(\mu)} = r_{(\mu)} \, sin\theta^{(\mu)} \, d\theta^{(\mu)} \wedge d\phi^{(\mu)}$$

and the moment map $J : T^*S^2 \longrightarrow so(3)^*$ associated with the Hall effect on the sphere S^2 can be expressed as $J^*(\bar{x}_i) = J_i$, $1 \leq i \leq 3$ and the energy of the system is given by

$$E = \frac{J_1^2 + J_2^2 + J_3^2 - \left(\frac{k}{2}\right)^2}{2mR^2} = J^*\left[\frac{\bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2 - \left(\frac{k}{2}\right)^2}{2mR^2}\right]$$

One has

Lemma. *If $k \neq 0$ (that is, there is a non null magnetic Hall field on the sphere), then*

1) If $r_{(\mu)} < \frac{k}{2}$ we have that $J^{-1}(\mathcal{O}_\mu) = \emptyset$.

2) If $r_{(\mu)} \geq \frac{k}{2}$ then $J^{-1}(\mathcal{O}_\mu)$ is the tridimensional submanifold of T^*S^2 of constant energy equal to $E_\mu = \frac{r_{(\mu)}^2 - \left(\frac{k}{2}\right)^2}{2mR^2}$.

Using the physical interpretation of the trajectories with angular momentum $\nu \in \mathcal{O}_\mu$ and the relationship between energy and angular momenta, it is easy to prove that

Theorem [13]. *If $k \neq 0$ and $r_{(\mu)} \geq \frac{k}{2}$ then $J^{-1}(\mathcal{O}_\mu)$ is a non trivial $SO(2)$-fibre bundle over a sphere of radius $d = \frac{kR}{2r_{(\mu)}^2}$.*

Theorem [13]. *$J : J^{-1}(\mathcal{O}_\mu) \longrightarrow \mathcal{O}_\mu$ is a non trivial $U(1)$-principal bundle.*

In order to endow the Hall effect on the sphere with a super-Poisson structure let us recall the following [3]: Let M and N be two symplectic manifolds with hamiltonian G actions on them, whose moment maps are ϕ_M and ϕ_N. If ω_N is the symplectic form of N, let us write N^- for the same manifold N, but with $-\omega_N$ as symplectic form. Then $M \times N^-$ is a hamiltonian G space with moment map $\phi_{M\times N^-}$ given by $\phi_{M\times N^-}(m,n) = \phi_M(m) - \phi_N(n)$.

Lemma. *Given $\mu = (\mu_1, \mu_2, \mu_3) \in so(3)^*$ such that $\mu_1^2 + \mu_2^2 + \mu_3^2 = r_{(\mu)}^2 \geq (\frac{k}{2})^2$, the $SO(3)$ action on the symplectic manifold $T^*S^2 \times \mathcal{O}_\mu^-$ has a moment map $\Psi : T^*S^2 \times \mathcal{O}_\mu^- \longrightarrow so(3)^*$ which, using spherical coordinates on the coadjoint orbit $\mathcal{O}_\mu$, can be expressed as*

$$\Psi^*(\bar{x}_1) = \Psi_1 = J_1 - r_{(\mu)}\, sin\theta^{(\mu)}\, cos\phi^{(\mu)}$$
$$\Psi^*(\bar{x}_1) = \Psi_2 = J_2 - r_{(\mu)}\, sin\theta^{(\mu)}\, sin\phi^{(\mu)}$$
$$\Psi^*(\bar{x}_1) = \Psi_3 = J_3 - r_{(\mu)}\, cos\theta^{(\mu)}$$

Let us notice that $d\Psi_1 \wedge d\Psi_2 \wedge d\Psi_3 \neq 0$, therefore $0 \in so(3)^*$ is a regular value for Ψ. Then, $\Psi^{-1}(0)$ is a submanifold of $T^*S^2 \times \mathcal{O}_\mu^-$ of dimension 3 defined by the zeroes of the functions $\{\Psi_i\,,\, 1 \leq i \leq 3\}$ and it can be see as the tridimensional submanifold T^*S^2 of constant energy $E_\mu = \frac{r_{(\mu)}^2 - (\frac{k}{2})^2}{2mR^2}$.

Proposition. *Let $\forall \nu \in \mathcal{O}_\mu$ be with $r_{(\mu)}^2 \geq (\frac{k}{2})^2$, one has that $\frac{\Psi^{-1}(0)}{SO(3)}$ can be identified with $\frac{J^{-1}(\mu)}{SO(2)}$ and in particular $\frac{\Psi^{-1}(0)}{SO(3)} = \frac{J^{-1}(\mu)}{SO(2)}$.*

Since $\Psi^{-1}(0)$ is a coisotropic submanifold of $T^*S^2 \times \mathcal{O}_\mu^-$, the cohomological description of the Marsden-Weinstein reduction of $T^*S^2 \times \mathcal{O}_\mu^-$ by the $SO(3)$ action is a particular case of the ones studied by Kostant-Sternberg in [5]. In this case the BRST algebra is

$$\bar{\Omega}_{BRST} = C^\infty(T^*S^2 \times \mathcal{O}_\mu^-) \otimes \Omega^{\cdot} so(3) \otimes \Omega^{\cdot} so(3)^*$$

If we take a basis $< \rho_i,\ \forall i = 1,2,3 >$ on $so(3)$, let $< \eta_j,\ \forall j = 1,2,3 >$ be its dual basis. The BRST operator of this complex is given by $\bar{\delta}_{BRST} = \bar{\delta}_0 + \bar{\delta}_1$ where

$$\bar{\delta}_0(g) = 0$$
$$\bar{\delta}_0(\rho_i) = J_i - \bar{x}_i$$
$$\bar{\delta}_0(\eta_j) = 0$$

and

$$\bar{\delta}_1(g) = \{J_j - \bar{x}_j, g\}\, \eta_j$$
$$\bar{\delta}_1(\rho_i) = -\epsilon_{isr}\rho_r\eta_s$$
$$\bar{\delta}_1(\eta_j) = -\frac{1}{2}\epsilon_{rsj}\eta_r \wedge \eta_s$$

for $g \in C^\infty(T^*S^2 \times \mathcal{O}_\mu^-)$.

It is verified that the BRST zeroth cohomology is $H^0(\bar{\delta}_{BRST}) = C^\infty(\frac{\Psi^{-1}(0)}{SO(3)})$

Since $C^\infty(T^*S^2 \times \mathcal{O}_\mu^-)$ is a Poisson algebra, $\Omega^{\cdot} so(3) \otimes \Omega^{\cdot} so(3)^* = \Omega^{\cdot}(so(3) \oplus so(3)^*)$ has a super-Poisson structure by defining the superbrackets of the generators $\{\rho_i, \eta_j\} = \delta_{ij}$, $\{\rho_i, \rho_j\} = 0$, $\{\eta_i, \eta_j\} = 0$ and taking into account that the tensor product of two Poisson superalgebras also it is, we have that

Proposition. *The complex $\bar{\Omega}_{BRST}$ has a structure of Poisson superalgebra such that the BRST operator $\bar{\delta}_{BRST}$ is a Poisson derivation. Also there is an element $\bar{Q} = (J_j - \bar{x}_j)\eta_j - \frac{1}{2}\epsilon_{rsj}\rho_j \wedge \eta_r \wedge \eta_s \in \bar{\Omega}^1_{BRST}$, called the BRST charge associated with the complex $\bar{\Omega}_{BRST}$ that fulfills $\bar{\delta}_{BRST}(\bar{\Omega}^{p,q}) = \{\bar{Q}, \bar{\Omega}^{p,q}\}$ and $\bar{\delta}_{BRST}(\bar{Q}) = \{\bar{Q}, \bar{Q}\} = 0$.*

If we fix $\mu \in so(3)^*$, one can define a symplectic inmersion $i_\mu : T^*S^2 \longrightarrow T^*S^2 \times \mathcal{O}^-_\mu$; that is, there is a epimorphism of Poisson algebras $i^*_\mu : C^\infty(T^*S^2 \times \mathcal{O}^-_\mu) \longrightarrow C^\infty(T^*S^2)$ defined by

$$\begin{aligned} i^*_\mu(f) &= f \quad \forall f \in C^\infty(T^*S^2) \\ i^*_\mu(\bar{x}_i) &= \mu_i \end{aligned}$$

Moreover the Lie algebra $so(2)$ of the isotropy group of μ is included in a natural way in the Lie algebra $so(3)$ of the symmetry group as rotation around the μ-axis; that is, there is a morphism of algebras

$$\begin{aligned} \iota_\mu : so(2) &\longrightarrow so(3) \\ \alpha &\longrightarrow \frac{\alpha\, \omega_{(\mu)}}{\mu_1^2 + \mu_2^2 + \mu_3^2}\,(\mu_1 A_1 + \mu_2 A_2 + \mu_3 A_3) \end{aligned}$$

that induces a linear epimorphism $\iota^*_\mu : \Omega^{\cdot} so(3)^* \longrightarrow \Omega^{\cdot} so(2)^*$ such that $\iota^*_\mu(\eta_j) = \frac{\omega_{(\mu)}}{\mu_1^2+\mu_2^2+\mu_3^2}\,\mu_j \eta$. Then we have

Proposition. *$\Gamma_\mu = i^*_\mu \otimes (Id \otimes \iota^*_\mu) : C^\infty(T^*S^2 \times \mathcal{O}^-_\mu) \otimes \Omega^{\cdot} so(3) \otimes \Omega^{\cdot} so(3)^* \longrightarrow C^\infty(T^*S^2) \otimes \Omega^{\cdot} so(3) \otimes \Omega^{\cdot} so(2)^*$ is an epimorphism of algebras making commutative the following diagram*

$$\begin{array}{ccc} \bar{\Omega}_{BRST} & \xrightarrow{\Gamma_\mu} & \Omega_{BRST} \\ {\scriptstyle \bar{\delta}}\downarrow & & \downarrow{\scriptstyle \delta} \\ \bar{\Omega}_{BRST} & \xrightarrow{\Gamma_\mu} & \Omega_{BRST} \end{array}$$

that is, $\Gamma_\mu \circ \bar{\delta}_{BRST} = \delta_{BRST} \circ \Gamma_\mu$ and it induces a morphism between the zeroth cohomologies

$$\Gamma_\mu : H^0(\bar{\delta}_{BRST}) = C^\infty(\frac{\Psi^{-1}(0)}{SO(3)}) \longrightarrow H^0(\delta_{BRST}) = C^\infty(\frac{J^{-1}(\mu)}{SO(2)})$$

It confirms us that $(\Omega_{BRST} = C^\infty(T^*S^2) \otimes \Omega^{\cdot} so(3) \otimes \Omega^{\cdot} so(2)^*, \delta_{BRST})$ is the BRST algebra associated with the Hall effect on the sphere.

Now let us define a super-Poisson structure on $\Omega_{BRST} = C^\infty(T^*S^2) \otimes \Omega^{\cdot} so(3) \otimes \Omega^{\cdot} so(2)^*$ by imposing that Γ_μ is a super-Poisson morphism. Then we obtain that

$$\begin{aligned} \{f, g\} &= \{f, g\}_{T^*S^2} \quad \forall f, g \in C^\infty(T^*S^2) \\ 0 &= \{\rho_i, f\} = \{\eta, f\} \quad \forall f \in C^\infty(T^*S^2) \\ 0 &= \{\rho_i, \rho_j\} \\ 0 &= \{\eta, \eta\} \\ \delta_{ij} &= \frac{\omega_{(\mu)}}{\mu_1^2 + \mu_2^2 + \mu_3^2}\,\mu_i\,\{\eta, \rho_j\} \end{aligned}$$

However $Q = \Gamma_\mu(\bar{Q}) = \frac{\omega_{(\mu)}\,\alpha}{\mu_1^2+\mu_2^2+\mu_3^2}\,\mu_j(J_j - \mu_j)\eta$ is not a BRST charge for δ_{BRST} since

$$\{Q, f\} = \delta_{BRST}(f)$$
$$\{Q, \eta\} = \delta_{BRST}(\eta)$$
$$\{Q, \rho_i\} = (J_i - \mu_i) \neq \delta_{BRST}(f)$$

Finally we see that

$$\delta_{BRST} = \{Q,\ \} - \frac{\omega_{(\mu)}}{\mu_1^2+\mu_2^2+\mu_3^2}\,\mu_j\epsilon_{jik}\,\rho_k\,\eta\,\frac{\partial}{\partial\rho_i}$$

References

[1] J. E. Marsden and T. Ratiu: Introduction to Mechanics and Symmetry. Springer-Verlag. 1994.

[2] R. Abraham and J.E. Marsden: Foundations of Mechanics. Benjamin Cummings Publishing Company Inc., 1981.

[3] V. Guillemin and S. Sternberg: Symplectic Techniques in Physics. Cambridge University Press, 1984.

[4] A.D. Browning and D. McMullan: The Batalin, Fradkin and Vilkovisky formalism for higher-order theories. J. Math. Phys. 28, 2, (1987), 438-444.

[5] B. Kostant and S. Sternberg: Symplectic Reduction, B.R.S. Cohomology and Infinite dimensional Clifford algebras. Ann. of Phys. 176, (1987), 49-113.

[6] M. Dubois-Violette: Systèmes Dynamiques Contraints : L' Approche Homologique. Ann. Inst. Fourier 37, 4 (1987), 45-57.

[7] M. Henneaux and C. Teitelboim: BRST Cohomology in Classical Mechanics. Commun. Math. Phys. 115, (1988),213-230.

[8] J. Stasheff: Constrained Poisson Algebras and Strong Homotopy Representations. Bull.Am. Math. Soc. 19, (1988), 287-290.

[9] M. Henneaux: Classical Foundations of BRST Symmetry. Bibliopolis (Monographs and Textbooks in Physical Sciences Lecture Notes n^o 7) Napoli, 1988.

[10] J. Fisch, M. Henneaux, J. Stasheff and C. Teitelboim: Existence, Uniqueness and Cohomology of the Classical BRST Charge with Ghosts of Ghost. Commun. Math. Phys. 120, (1989),379-407.

[11] P. L. García: The Poincaré-Cartan Invariant in the Calculus of Variations. Sym.Mathematica 14,(1974), 219-246

[12] A. López Almorox and C. Tejero Prieto: Infinitesimal Symmetries of the Hall Effect on Riemann Surfaces. In this Conference.

[13] A. López Almorox and C. Tejero Prieto: Geometrical aspects of the Hall effect on the sphere. In preparation.

[14] G.V. Dunne: Hilbert Space for Charged Particles in Perpendicular Magnetic Fields. Ann. of Phys. 215, (1992), 233-263.

[15] I. Kogan: Area-Preserving Diffeomorphisms, W_∞ and $U_q(sl(2))$ in Chern-Simons Theory and Quantum Hall System. Preprint 1994 hep-th/9401093.

Departamento de Matemática Pura y Aplicada, Universidad de Salamanca
Plaza de la Merced 1-4, 37008 Salamanca, Spain
E-mail: alm440@rs6000.usal.es, carlost@gugu.usal.es