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A GEOMETRIC DEFINITION OF LIE DERIVATIVE FOR SPINOR FIELDS

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ABSTRACT. Relying on the general theory of Lie derivatives a new geometric definition of Lie derivative for general spinor fields is given, more general than Kosmann's one. It is shown that for particular infinitesimal lifts, i.e. for Kosmann vector fields, our definition coincides with the definition given by Kosmann more than 20 years ago.

1 Introduction

It is known that starting from some works by physicists (Dirac [1,2], Pauli [3]) the notion of spinor was introduced in Physics, although its discovery is due to E. Cartan in 1913 (cf. [4]) in his researches on the linear representations of some simple Lie groups.
A pretty nice and detailed algebraic theory was given later by C. Chevalley [5] and other papers on Clifford–algebras and spinor groups were published, in particular by J. Dieudonné [6], Atiyah, Bott, Shapiro [7], Milnor [8].

Spinors on curved space–time manifolds were introduced and used since 1928 by physicists. In a series of very important papers Yvette Kosmann [9,10,11,12] introduced the notion of Lie derivative for spinor fields on spin manifolds and the study of the problem of transforming a spinor field under one–parameter group of diffeomorphisms of the base manifold.

However, thanks to her beautiful and relevant work, we have the feeling that Kosmann's theory was not formulated in the clearest possible way and that the geometric meaning of her *"ad hoc"* definition was still lacking. In fact, it is quite known that the difficulty in defining the Lie derivative of a spinor field on a space (and time) oriented (pseudo) Riemannian manifold (M, g) comes from the fact that there is no natural definition of the image of such a spinor field by a diffeomorphism. This corresponds to the fact that the bundle $SO(M, g)$ together with any spin bundle $Spin(M, g)$ covering $SO(M, g)$ are not natural bundles [17].

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Of course, these bundles are particular gauge–natural bundles [17,21] and we will show moreover that it is always possible to lift in a unique way any vetor field $\xi \in \mathfrak{X}(M)$ of the base manifold (M, g).

The corresponding vector field $\tilde{\xi}_K \in \mathfrak{X}(Spin(M, g))$, projectable over ξ, and $\xi_K \in \mathfrak{X}(SO(M, g))$ also projectable over ξ, will be called the Kosmann vector fields of $\xi \in \mathfrak{X}(M)$. We remark that the vector field $\tilde{\xi}_K$ also projects over ξ_K.

The Kosmann lifting is not natural and differs from the Levi–Civita lifting, i.e. the principal lifting induced by the Levi–Civita connection on $SO(M, g)$ (first order derivatives of ξ are involved), although the Ricci rotation coefficients of the Levi–Civita connection appear in the expression which gives the Kosmann lifting of ξ.

In this paper we shall give a *new* definition of Lie derivative for spinor fields and we will show that for particular infinitesimal lifts, i.e. for Kosmann vector fields, our definition coincides with the definition given by Kosmann more than 20 years ago.

2 Spin structures

Let (P, M, π, G) be a principal fiber bundle over M with structural group a Lie group G. Let $\rho\colon \Gamma \to G$ be a central homomorphism of a Lie group Γ onto G with kernel K, that is K is discrete and contained in the center of Γ (see Greub and Petry in [13]; see also [22]). We recall that a Γ*–structure* on P is a principal bundle map $\eta\colon \tilde{P} \to P$ which is equivariant under the right actions of the structure groups, that is:

$$\eta(\tilde{z} \cdot \gamma) = \eta(\tilde{z}) \cdot \rho(\gamma) \quad , \tilde{z} \in \tilde{P},\ \gamma \in \Gamma\,, \tag{2.1}$$

and where ρ is assumed central by definition.
Equivalently the following diagram commutes:

$$\begin{array}{ccc} \tilde{P} \times \Gamma & \xrightarrow{\eta \times \rho} & P \times G \\ \downarrow{\scriptstyle \tilde{R}} & & \downarrow{\scriptstyle R} \\ \tilde{P} & \xrightarrow{\eta} & P \\ \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{id_M} & M \end{array} \tag{2.2}$$

where R and $\tilde{R}$ denote right multipication in P and $\tilde{P}$ respectively .
This means that for $\tilde{z} \in \tilde{P}$, both $\tilde{z}$ and $\eta(\tilde{z})$ lie over the same point, and that η restricted to a fiber is a "copy" of ρ, i.e. equivalent to ρ.

In general, it is not guaranteed that a manifold admits a spin structure. The existence condition for a Γ–structure on P can be formulated [13,14,22] in terms of Čech cohomology.

Let us also recall that for any principal fiber bundle (P, M, π, G) an *(principal) automorphism* of P is a diffeomorphism $\phi\colon P \to P$ such that $\phi(u \cdot g) = \phi(u) \cdot g$, for every $u \in P$, $g \in G$. Each ϕ induces a unique diffeomorphism $\varphi\colon M \to M$ such that $\pi \circ \phi = \varphi \circ \pi$. Accordingly, we denote by $Aut(P)$ the group of all principal (i.e. equivariant) automorphisms of P. Assume a vector field $X \in \mathfrak{X}(P)$ on P generates the one–parameter group ϕ_t. Then, X is G–invariant if and only if ϕ_t is an automorphism of P for every $t \in \mathbb{R}$. Accordingly, we denote by $\mathfrak{X}_G(P)$ the Lie algebra of G–invariant vector fields of P.

Let $\mathbb{R}^{p,q}$ be the familiar vector space $\mathbb{R}^n$ $(p+q=n)$ equipped with the canonical nondegenerate symmetric bilinear form $<,>$ of signature (p,q), i.e. with p "+" signs and with q "-" signs. We will denote by $O(p,q)$ the (pseudo)–orthogonal group with respect to $<,>$, that is $O(p,q)=\{L\in Gl(\mathbb{R}^{p,q})|<Lu,Lv>=<u,v>\}$, by $SO(p,q)=\{L\in O(p,q)|\det(L)=1\}$ its special subgroup, and by $SO_0(p,q)$ its connected component with identity. In the Euclidean case (p or $q=0$), $SO(p,q)=SO_0(p,q)$ but in the general case they are not equal. To define the Clifford algebra $Cl(p,q)\equiv Cl(\mathbb{R}^{p,q})$ we choose any orthonormal basis e_a of $\mathbb{R}^{p,q}\subset Cl(p,q)$, that is $<e_a,e_b>=\eta_{ab}$. The Clifford algebra $Cl(p,q)$ is the real vector space endowed with an associative product generated (as an algebra) by the unit I and the elements e_a, $1\le a\le n=p+q$, with the relations:

$$e_a e_b + e_b e_a = -2\eta_{ab} I \tag{2.3}$$

where $\eta_{ab}=0$ if $a\neq b$, $+1$ if $a=b\le p$, and -1 if $p<a=b$. As a real vector space, the Clifford algebra $Cl(p,q)$ has dimension 2^n.

The set $\{I, e_a, e_{a_1}e_{a_2}, ..., e_{a_1}e_{a_2}\cdots e_{a_r}, e_{a_1}\cdots e_{a_n} \,:\, a_1<a_2<\cdots<a_r\}$ forms a basis of $Cl(p,q)$. The complexified Clifford algebra $\mathbb{C}l(p,q)$ is the vector space $Cl(p,q)\otimes_{\mathbb{R}}\mathbb{C}$, that is the vector space over the complex numbers generated by the previous elements $I, e_a, ..., e_{a_1}\cdots e_{a_n}$. In this paper we shall assume that the dimension of $\mathbb{R}^n$ is even, that is $n=2d$. In this case it can be proved that $\mathbb{C}l(p,q)$ is isomorphic to the complex algebra $M_{2^d}(\mathbb{C})$ of complex matrices of order 2^d. The matrices representing the fundamental elements e_a will be denoted by γ_a. They automatically carry a representation of various subgroups of $Cl(p,q)$ and in particular of $Spin(p,q)$.

Let us briefly recall how $Spin(p,q)$ is defined. Consider the multiplicative group of units of $Cl(p,q)$, which is defined to be the subset

$$Cl^{\times}(p,q)\equiv\{a\in Cl(p,q)\,:\,\exists a^{-1} \text{ such that } a^{-1}a=aa^{-1}=1\} \quad . \tag{2.4}$$

This group contains all elements $v\in\mathbb{R}^{p,q}\subset Cl(p,q)$ with $<v,v>\neq 0$ so it is possible to consider the subgroup generated by its "units length" elements, $v\in\mathbb{R}^{p,q}\subset Cl(p,q)$ with $<v,v>=\pm1$.

This subgroup it is denoted by $Pin(p,q)$. The spin group $Spin(p,q)$ is defined as the subgroup of $Pin(p,q)$ consisting of even elements:

$$Spin(p,q)=Pin(p,q)\cap Cl^0(p,q) \quad , \tag{2.5}$$

where $Cl^0(p,q)$ is a subalgebra of $Cl(p,q)$ called the *even part* of $Cl(p,q)$.

Now, consider a space (and time) oriented (pseudo)–Riemannian manifold with metric g and dimension $n=dim(M)$. In this case we have $G=SO(p,q)$, $\Gamma=Spin(p,q)$, $P=SO(M,g)\subset L(M)$, $\tilde{P}=Spin(M,g)$ where $Spin(p,q)$ is the usual notation for $Spin(\mathbb{R}^{p,q})$ and $(L(M),Gl(n),M)$ is the principal bundle of linear frames. We shall say that a *spin–structure* on a $SO(p,q)$–bundle is a $Spin(p,q)$–structure with respect to the covering of Lie groups $\rho\colon Spin(p,q)\to SO(p,q)$ (see Milnor in [8]). For more details in Clifford–algebras and spinor groups see also [7,14,15,16]. When a spin structure is given $\eta\colon Spin(M,g)\to SO(M,g)$, the principal bundle $Spin(M,g)$ is sometimes called the *bundle of spinor frames*.

Let us also recall that, in this case, the local isomorphism between $Spin(M,g)$ and $SO(M,g)$ allows the Levi–Civita connection to be lifted to a connection on the principal bundle $Spin(M,g)$, equivariant by $Spin(p,q)$, $p+q=n$.
Let $\omega\colon T[SO(M,g)] \to \mathfrak{so}(p,q)$ be the connection one–form determined by the Levi–Civita connection of (M,g). The corresponding connection σ on $Spin(M,g)$

$$\sigma\colon T[Spin(M,g)] \to \mathfrak{spin}(p,q)$$

is defined by the composition:

$$\sigma = (\rho')^{-1} \circ (\eta^* \omega) \tag{2.6}$$

where $\rho' = T_e\rho\colon \mathfrak{spin}(p,q) \to \mathfrak{so}(p,q)$ is the Lie algebras isomorphism induced by ρ.

Spinors are defined via a linear representation of the spin group $Spin(p,q)$ on a complex vector space S, with $dim_C S = 2^d$.
In fact, let $\chi\colon Spin(p,q)\times S \to S$ be a *spinor* representation:

$$\chi\colon Spin(p,q)\times S \to S : (h,s) \mapsto \chi(h,s) \equiv h\cdot s \quad . \tag{2.7}$$

We recall that any representation of $SO(p,q)$ yields a representation of its covering group, just by composition with ρ. We shall call a representation of this kind a *tensor representation* of $Spin(p,q)$. A *spinor representation* of $Spin(p,q)$ (sometimes called a double representation of $SO(p,q)$) is by definition a linear representation of $Spin(p,q)$ which cannot be obtained from a representation of $SO(p,q)$.

Finally, consider the vector bundle $S(M) \equiv Spin(M,g)\times \mathbb{C}^d / Spin(p,q)$ over M associated to the principal fiber bundle $Spin(M,g)$, where the action of $Spin(p,q)$ on $\mathbb{C}^d$ is the fundamental one defined by inclusion of $Spin(p,q)$ in the complexified Clifford algebra $\mathbb{C}l(p,q)$. Then a spinor field ψ is a section $x \mapsto [\tilde e_x, \Psi(\tilde e_x)]$ of $S(M)$, where $\Psi\colon Spin(M,g) \to \mathbb{C}^d$, $\Psi(\tilde e_x \cdot h) = h^{-1}\cdot \Psi(\tilde e_x)$, is an equivariant function, i.e. an assignment of components $\Psi(\tilde e_x)$ in $\mathbb{C}^d$ to each "spinor frame" $\tilde e_x$.

3 Lie derivative of Spinor Fields

Given a spin structure $Spin(M,g)\to M$, a *generalized spinorial transformation* Φ of $Spin(M,g)$ is an automorphism $\Phi\colon Spin(M,g) \to Spin(M,g)$. As discussed above Φ is a diffeomorphism of $Spin(M,g)$ such that $\Phi(\tilde e\cdot h) = \Phi(\tilde e)\cdot h$, for every $\tilde e \in Spin(M,g)$, $h \in Spin(p,q)$. Each Φ induces a unique diffeomorphism $\varphi\colon M \to M$ such that $\eta_M \circ \Phi = \varphi \circ \eta_M$, with $\eta_M = \pi\circ\eta$. Moreover, in this case, Φ also induces a unique automorphism (i.e. equivariant diffeomorphism) $\phi\colon SO(M,g) \to SO(M,g)$ such that $\eta\circ\Phi = \phi\circ\eta$.
Equivalently we have the following diagram:

$$\begin{array}{ccc} Spin(M,g) & \xrightarrow{\Phi} & Spin(M,g) \\ \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta} \\ SO(M,g) & \xrightarrow{\phi} & SO(M,g) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow{\varphi} & M \end{array} \tag{3.1}$$

Furthermore, let $S(M)$ be the fiber bundle of spinors associated to the principal bundle $Spin(M,g)$. For every element $\Phi \in Aut(Spin(M,g))$ we obtain an automorphism $\Phi_{S(M)}$ of $S(M)$, $\Phi_{S(M)}\colon S(M) \to S(M)$, given by $\Phi_{S(M)}(z) = [\Phi(\tilde{e}), s]$, $z = [\tilde{e}, s] \in S(M)$, which is well defined because it does not depend on the representative chosen. In fact, this is so since Φ is equivariant under the right action of the structure group $Spin(p,q)$. A vector field $\tilde{\Xi} \in \mathfrak{X}(Spin(M,g))$ generating a one–parameter group of automorphisms of $Spin(M,g)$ defines the generalized Lie derivative of any section $\psi\colon M \to S(M)$, that is of any spinor field, as follows:

$$\tilde{\mathcal{L}}_{\tilde{\Xi}}\psi = T\psi \circ \xi - \tilde{\Xi}_{S(M)} \circ \psi \quad , \tag{3.2}$$

where $\xi \in \mathfrak{X}(M)$ is the only vector field such that $T\pi \circ \Xi = \xi \circ \pi$, Ξ is the only vector field such that $T\eta \circ \tilde{\Xi} = \Xi \circ \eta$, and for $\tilde{\Xi}_{S(M)}$ we have: $\tilde{\Xi}_{S(M)}(z) = \frac{d}{dt}[(\Phi_t)_{S(M)}(z)]|_{t=0}$, for every $z \in S(M)$.
Once again we have the following diagram:

$$\begin{array}{ccc}
Spin(M,g) & \xrightarrow{\tilde{\Xi}} & T[Spin(M,g)] \\
\Big\downarrow{\scriptstyle \eta} & & \Big\downarrow{\scriptstyle T\eta} \\
SO(M,g) & \xrightarrow{\Xi} & T[SO(M,g)] \\
\Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle T\pi} \\
M & \xrightarrow{\xi} & TM
\end{array} \tag{3.3}$$

Hence *the generalized Lie derivative of* ψ *with respect to* $\tilde{\Xi}$, $\tilde{\mathcal{L}}_{\tilde{\Xi}}\psi\colon M \to V[S(M)]$, takes values in the vertical sub-bundle $V[S(M)] \subset T[S(M)]$ and being $S(M)$ a vector bundle there is a canonical identification $V[S(M)] = S(M) \oplus S(M)$. In this case, (3.2) is of the form $\tilde{\mathcal{L}}_{\tilde{\Xi}}\psi = (\psi, \mathcal{L}_{\tilde{\Xi}}\psi)$.
The first component being the original section ψ, the second component $\mathcal{L}_{\tilde{\Xi}}\psi$ is also a section of $S(M)$, and it is called the *Lie derivative of* ψ *with respect to* $\tilde{\Xi}$. Sometimes, the second component $\mathcal{L}_{\tilde{\Xi}}\psi$ is called the *restricted Lie derivative* [17]. Using the fact that the second component of $\tilde{\mathcal{L}}_{\tilde{\Xi}}\psi$ is the derivative of $(\Phi_{-t})_{S(M)} \circ \psi \circ \varphi_t$ for $t = 0$ in the classical sense, one can re-express the restricted Lie derivative in the form

$$(\mathcal{L}_{\tilde{\Xi}}\psi)(x) = \lim_{t\to 0} \frac{1}{t}((\Phi_{-t})_{S(M)} \circ \psi \circ \varphi_t(x) - \psi(x)) \quad . \tag{3.4}$$

Further details and deeper discussions on the general theory of Lie derivatives may be found in [17,18,19].

We are now seeking a coordinate expression for the restricted Lie derivative given by the equation (3.4). Now, if (x^λ, ψ^i) denote local fibered coordinates for $S(M)$ then (3.4) reads as:

$$\mathcal{L}_{\tilde{\Xi}}\psi = [\xi^a e_a \psi^i - \tilde{\Xi}^i_j \psi^j] f_i \quad , \tag{3.5}$$

where $\xi = \xi^a(x)e_a$, $e = (e_a) = (e^\mu_a(x)\partial_\mu)$ is a local section of $SO(M,g)$ induced by a local section of $Spin(M,g)$ $\tilde{e}\colon U \to Spin(M,g)$ such that $\eta \circ \tilde{e} = e$, $e_a(\psi^i)$ is the Pfaff derivative of ψ^i, that is $e_a\psi^i = e^\mu_a(x)\partial_\mu\psi^i$ and f_i is a basis of $[S(M)]_x$ at each $x \in U$.

Recalling that $\tilde{\Xi}$ projects on Ξ and taking into account the Lie algebras isomorphism $\rho' = T_e\rho \colon \mathfrak{spin}(p,q) \to \mathfrak{so}(p,q)$, we find the following relations:

$$\tilde{\Xi}^i_j = -\frac{1}{4}\Xi^a{}_b(\gamma_a\gamma^b)^i_j = -\frac{1}{4}\Xi_{ab}(\gamma^a\gamma^b)^i_j \tag{3.6a}$$

$$\tilde{\Xi}^a = \Xi^a = \xi^a \tag{3.6b}$$

where indices in (3.6a) are lowered and raised with respect to the (pseudo) Riemannian metric g, i.e. $\eta_{ab} = g(e_a, e_b)$. Recall that for Ξ, being a $SO(p,q)$-invariant vector field of $SO(M,g)$, in the orthonormal basis $(e^\mu_a(x)\partial_\mu)$, we have $\Xi_{ab} = -\Xi_{ba}$. We also note that the sign in (3.6a) depends on the choice of sign for the Clifford algebra, and the choice between $L^a{}_b\gamma_a$ and $L_b{}^a\gamma_a$, where $L^a{}_b$ and $L_b{}^a$ are any matrices.

Finally, for the spin connection coefficients we have:

$$\sigma^i_{jc} = -\frac{1}{4}\omega^a{}_{bc}(\gamma_a\gamma^b)^i_j = -\frac{1}{4}\omega_{abc}(\gamma^a\gamma^b)^i_j \quad . \tag{3.7}$$

So we can re–write the (3.5) as follows:

$$\mathcal{L}_{\tilde{\Xi}}\psi = \{\xi^a\nabla_a\psi^i - [-\frac{1}{4}(\omega_{[ab]c}\xi^c + \Xi_{ab})(\gamma^a\gamma^b)^i_j\psi^j]\}f_i \quad , \tag{3.8}$$

where in the orthonormal frame $(e^\mu_a(x)\partial_\mu)$ we have $\omega_{abc} = -\omega_{bac}$ and square brackets on indices mean complete antisymmetrization, although in this case $\omega_{[ab]c} = \omega_{abc}$. In fact, the coefficients ω_{abc} are the Ricci rotation coefficients defined by:

$$\omega_{abc} = \eta_{ad}\omega^d{}_{bc} \quad . \tag{3.9}$$

Moreover, for $\nabla_a\psi^i$ we have the local expression:

$$\nabla_a\psi^i = e_a\psi^i + \sigma^i_{ja}\psi^j \quad . \tag{3.10}$$

In this notation we have, in the same orthonormal basis $(e^\mu_a(x)\partial_\mu)$, the following: $\nabla_{e_a}e_b = \omega^c{}_{ba}e_c$ for the connection coefficients and $\omega^a{}_b = \omega^a{}_{bc}(x)\theta^c$ for the connection one-forms, where θ^c is the dual basis of e_a. Hence, in our notation we get for the covariant derivative $\nabla_a\xi_b$ the following local expression:

$$\nabla_a\xi_b = e_a\xi_b - \omega^c{}_{ba}\xi_c = e_a\xi_b - \omega_{cba}\xi^c \quad , \tag{3.11}$$

where $\xi_c = \eta_{cb}\xi^b$.

Given a space (and time) oriented (pseudo) Riemannian manifold (M,g) and a vector field ξ it is possible to give its lift to $SO(M,g)$, denoted by ξ_K. The vector field ξ_K has the following coordinated expression:

$$\xi_K = (\xi_K)^a e_a + (\xi_K)_{ab}A^{ab} \quad , \tag{3.12}$$

where the coefficients $(\xi_K)^a$ and $(\xi_K)_{ab}$ are given by:

$$(\xi_K)^a = \xi^a \qquad (\xi_K)_{ab} = -\{\nabla_{[a}\xi_{b]} + \omega_{[ab]c}\xi^c\} \quad , \tag{3.13}$$

and where $\nabla_{[a}\xi_{b]} = e_{[a}\xi_{b]} - \omega_{c\,[ba]}\xi^c$. The vector fields A^{ab} are local right $SO(p,q)$–invariant vector fields on $SO(M,g)$ and in a suitable chart (x^μ, u^b_a) are defined as follows:

$$A^{ab} = \frac{1}{2}(\eta^{ac}\delta^b_d - \eta^{bc}\delta^a_d)u^d_e\frac{\partial}{\partial u^c_e} \quad . \tag{3.14}$$

In order to define local coordinates (x^μ, u^b_a) which appear in (3.14) we remind that, as above, the local section $e: U \to SO(M,g)$ is induced by a local section of $Spin(M,g)$ $\tilde{e}: U \to Spin(M,g)$, so that we have $\eta \circ \tilde{e} = e$. Moreover, if (x^μ) are local coordinates of a chart of the base manifold M with the same domain U, we can define local coordinates u^μ_a by $(u^\mu_a(u)\partial_\mu)$ where $u = (u_a)$ is any frame in the open $\pi^{-1}(U)$ of $L(M)$ such that $g(u_a, u_b) = \eta_{ab}$. Finally, with the local section $e: U \to SO(M,g)$ we define local coordinates u^b_a by $u_a = u^b_a e_b$.

Accordingly, ξ_K transforms as a $SO(p,q)$–invariant vector field on $SO(M,g)$ and projects over ξ. An alternative geometric definition of ξ_K is the following one. Given any vector field ξ of the base manifold M,it admits a unique natural lift $\hat{\xi}$ to the linear frame bundle $L(M)$. Then, ξ_K is by definition the skew of $\hat{\xi}$ with respect to the (pseudo) Riemannian metric g.

Furthermore, when a spin structure is given $\eta: Spin(M,g) \to SO(M,g)$, any vector field ξ lifts to a vector field on the spin bundle $Spin(M,g)$. In fact, its lift $\tilde{\xi}_K$ is defined by:

$$\tilde{\xi}_K = (\tilde{\xi}_K)^a e_a + (\tilde{\xi}_K)^i_j E^j_i \quad , \tag{3.15}$$

and the coefficients $(\tilde{\xi}_K)^a$, $(\tilde{\xi}_K)^i_j$ are given by:

$$(\tilde{\xi}_K)^a = \xi^a \qquad (\tilde{\xi}_K)^i_j = -\frac{1}{4}(\xi_K)_{ab}(\gamma^a\gamma^b)^i_j \quad , \tag{3.16}$$

so that $\tilde{\xi}_K$ projects over ξ_K and hence over ξ. Accordingly, the following diagram commutes:

$$\begin{array}{ccc} Spin(M,g) & \xrightarrow{\tilde{\xi}_K} & T[Spin(M,g)] \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle T\eta} \\ SO(M,g) & \xrightarrow{\xi_K} & T[SO(M,g)] \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle T\pi} \\ M & \xrightarrow{\xi} & TM \end{array} \tag{3.17}$$

The vector field $\tilde{\xi}_K$ is well defined because the one–parameter group of automorphisms $\{\phi_t\}$ of ξ_K can be lifted (see below) to an one–parameter group of automorphisms $\{\tilde{\phi}_t\}$ so that $\tilde{\xi}_K(\tilde{e}) = \frac{d}{dt}[\tilde{\phi}_t(\tilde{e})]|_{t=0}$, for every $\tilde{e} \in Spin(M,g)$.

The vector fields $\tilde{\xi}_K$, ξ_K, invariant under the actions of $Spin(p,q)$ and $SO(p,q)$, are called the *Kosmann vector fields* of the vector field ξ. Let us also remark that the Kosmann lifting $\xi \longmapsto \xi_K$ is not a Lie–algebra homomorphism although the following remarkable relation holds:

$$[\xi,\zeta]_K = [\xi_K, \zeta_K] - \frac{1}{2}\mathcal{L}_\xi(g_{\lambda\mu})g^{\mu\nu}\mathcal{L}_\zeta(g_{\nu\sigma})A^{\lambda\sigma} \tag{3.18a}$$

where

$$A^{\lambda\sigma} = e^\lambda_a e^\sigma_b A^{ab} \quad . \tag{3.18b}$$

We are now in a good position to state the following result.

Proposition. *The Lie derivative $\mathcal{L}_\xi\psi$ of a spinor field $\psi: M \to S(M)$ with respect to a vector field $\xi \in \mathfrak{X}(M)$, as defined by Kosmann in [9], is the second component* *of the generalized Lie derivative of ψ with respect to the Kosmann lift $\tilde{\xi}_K$ of ξ. That is if $\tilde{\mathcal{L}}_{\tilde{\xi}_K}\psi$ is the generalized Lie derivative of ψ and $\mathcal{L}_{\tilde{\xi}_K}\psi$ its second component then we get $\mathcal{L}_\xi\psi = \mathcal{L}_{\tilde{\xi}_K}\psi$.*

Proof: It is an immediate consequence of the above considerations and from the fact that the local expression of the Lie derivative given by Kosmann is

$$\mathcal{L}_\xi\psi = [\xi^a\nabla_a\psi^i - \frac{1}{4}\nabla_{[a}\xi_{b]}(\gamma^a\gamma^b)^i_j\psi^j]f_i \quad . \tag{3.19}$$

(Q.E.D)

We remark that the above proposition gives us a geometric meaning to the Kosmann's formula (3.19) which is taken by Kosmann as a definition of the Lie derivative of ψ with respect to ξ. We shall use the following notation: $\tilde{\mathcal{L}}_\xi\psi := \tilde{\mathcal{L}}_{\tilde{\xi}_K}\psi$. We finally remark that there is an easier interpretation of the above facts in the case in which ξ is an infinitesimal isometry, i.e. a Killing vector field on M.

In fact, given a spin structure η: $Spin(M,g) \to SO(M,g)$, let us consider a Killing vector field ξ, its flow denoted by $\{\varphi_t\}$ and the principal bundle map ϕ_t: $SO(M,g) \to SO(M,g)$ which is defined by restriction of $L_{\varphi_t}: L(M) \to L(M)$ to $SO(M,g)$. The automorphisms ϕ_t are well defined since the diffeomorphisms φ_t are isometries. Moreover being η: $Spin(M,g) \to SO(M,g)$ a covering space it is possible to lift ϕ_t to a bundle map $\tilde{\phi}_t$: $Spin(M,g) \to Spin(M,g)$ in the following way. For any spinor frame $\tilde{e} \in Spin(M,g)$ over $\eta(\tilde{e}) = e$, from the theory of covering spaces it follows that for the curve $\gamma_e: \mathbb{R} \to SO(M,g)$, based at e, that is $\gamma_e(0) = e$, and defined by $\gamma_e(t) := \phi_t(e)$ there exists a unique curve $\tilde{\gamma}_{\tilde{e}}: \mathbb{R} \to Spin(M,g)$, based at $\tilde{e}$, such that $\eta \circ \tilde{\gamma}_{\tilde{e}} = \gamma_e$. It is possible to define a principal bundle map $\tilde{\phi}_t$: $Spin(M,g) \to Spin(M,g)$, covering ϕ_t, by letting $\tilde{\phi}_t(\tilde{e}) := \tilde{\gamma}_{\tilde{e}}(t)$. The one–parameter group of automorphisms of $Spin(M,g)$ $\{\tilde{\phi}_t\}$ defines a vector field $\tilde{\Xi}(\tilde{e}) = \frac{d}{dt}[\tilde{\phi}_t(\tilde{e})]|_{t=0}$, for all $\tilde{e} \in Spin(M,g)$.

It is then easy to verify that the vector field $\tilde{\Xi}$ coincides with the Kosmann vector field $\tilde{\xi}_K$ of the vector field ξ. In fact, in this case, the lift $\hat{\xi}$ is already skewsymmetric. Hence the Lie derivative of ψ with respect to ξ is still given by the Kosmann's formula (3.19) (cf. [23,24]) or, better, by the following coordinate–free expression

$$(\mathcal{L}_\xi\psi)(x) = \lim_{t\to 0}\frac{1}{t}((\tilde{\phi}_{-t})_{S(M)} \circ \psi \circ \varphi_t(x) - \psi(x)) \quad . \tag{3.20}$$

Moreover, even for an arbitrary vector field ξ the formula (3.20) is still valid but where now $\{\phi_t\}$ is the flow of the vector field ξ_K, and $\{\tilde{\phi}_t\}$ is the flow of $\tilde{\xi}_K$. Now, ξ is not an infinitesimal isometry but ξ_K generates a one–parameter group of principal automorphisms of $SO(M,g)$ which lift to $Spin(M,g)$ so that even in this case (3.20) gives us a further interpretation of Kosmann's formula, i.e. a direct classical definition of the Lie derivative for spinor fields.

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