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POISSON MANIFOLDS OF GEODESIC ARCS

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ABSTRACT. In the present paper a notion of a Poisson manifold of geodesic arcs is introduced and studied. Local coordinate expressions are found. A relation between Poisson manifolds of geodesic arcs and the classical Lagrangian mechanics is clarified.

Definitions of fundamental notions

Let us consider a smooth and finite-dimensional manifold X. A *Poisson algebra over* X is a Lie algebra structure $\{\ ,\ \}$ on $C^\infty(X)$ that satisfies the *Leibniz condition* $\{FG, H\} = F\{G, H\} + G\{F, H\}$. A *Poisson bracket* $\{\ ,\ \}$ is the Lie bracket of the corresponding Poisson algebra. A *Casimir function* in a Poisson algebra over X is a function $F \in C^\infty(X)$ such that $\{F, G\} = 0$ for all functions $G \in C^\infty(X)$. A *Poisson manifold* is a smooth manifold X with a Poisson algebra over X (Lichnerowicz [1], Weinstein [2], and others). A *Poisson mapping* is a homomorphism of Poisson manifolds. A *Poisson submanifold* is a submanifold Y in a Poisson manifold X with a Poisson algebra for which the inclusion is a Poisson mapping. Note that any open submanifold in a Poisson manifold is the Poisson submanifold. A Poisson manifold is *Abelian* if and only if the corresponding Poisson algebra is Abelian. A Poisson manifold is a *symplectic manifold* if and only if each of its Casimir functions is a constant function.

Let us consider a symmetric linear connection Γ on X and a closed interval $\langle 0, 1\rangle \subset \mathbb{R}$. A *geodesic arc* γ of the connection Γ is a smooth mapping $\gamma : \langle 0, 1\rangle \to X$ that is a geodesic of connection Γ. Let $W(X)$ be a set of all geodesic arcs of the connection Γ. It is known (see, e.g., Kobayashi and Nomizu [3]) that there exists a bijective mapping

$$\beta : W(X) \ni \gamma \to \dot\gamma(0) \in V, \tag{1}$$

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where $\dot{\gamma} : \langle 0, 1\rangle \to TX$ is the prolongation of geodesic arc γ on the tangent bundle TX and V is an open subset of TX. The set $W(X)$ equipped with a structure of smooth manifold such that β is a diffeomorphism is called a *manifold of geodesic arcs*.

Let us consider two manifolds of geodesic arcs $W(X)$ and $W(Y)$. Any smooth mapping $f : X \to Y$ such that $\gamma \in W(X)$ implies $f \circ \gamma \in W(Y)$ is called a *geodesic mapping*. The geodesic mapping f defines a mapping $L_f : W(X) \ni \gamma \to f \circ \gamma \in W(Y)$ of manifolds of geodesic arcs. Evidently, if X is a fibred manifold over the basis Y with the projection f then $W(X)$ is a fibred manifold over the basis $W(Y)$ with projection L_f. In such a case $W(X)$ is called a *fibred manifold of geodesic arcs* $L_f : W(X) \to W(Y)$. Similarly, if Y is a submanifold of the manifold X with the inclusion f then $W(Y)$ is a submanifold of $W(X)$ with the inclusion L_f. In such a case Y is called a *geodesic submanifold of the manifold* X.

Let us consider a monoid M of all polynomial mappings $\langle 0, 1\rangle \to \langle 0, 1\rangle$ whose degree is less than two with the multiplication $M \times M \ni (\mu, \nu) \to \mu \circ \nu \in M$. The global chart $M \ni \mu \to (\mu(0), \mu(1)) \in \langle 0, 1\rangle \times \langle 0, 1\rangle$ defines a structure of a smooth manifold with corners on M in such a way that the multiplication is smooth. The monoid M with the above defined structures will be called the *Lie monoid* M. Let $W(X)$ is a manifold of geodesic arcs. It is known (see, e.g., [3], [5]) that $\gamma \in W(X)$, $\mu \in M$ implies $\gamma \circ \mu \in W(X)$ and so the Lie monoid M acts on $W(X)$ from right. Since a mapping $V \times \langle 0, 1\rangle \ni (a, b) \to \beta^{-1}(a)(b) \in X$ is smooth, we see that the action $R : W(X) \times M \ni (\gamma, \mu) \to \gamma \circ \mu \in W(X)$ is smooth too. We shall use a partial mapping

$$R_\mu : W(X) \ni \gamma \to \gamma \circ \mu \in W(X). \tag{2}$$

The action R of the Lie monoid M is of fundamental importance in the theory of geodesic arcs. It can be shown (Whitehead [4], Klapka [5]) that the notion of a symmetric linear connection may be defined with the help of (2).

Definition. *Let $W(X)$ be a manifold of geodesic arcs. A Poisson right M-algebra over $W(X)$ is a Poisson algebra over $W(X)$ such that for each $\mu \in M$ a mapping $C^\infty(W(X)) \ni \sqrt{F} \circ R_\mu \in C^\infty(W(X))$ is an endomorphism. A Poisson manifold of geodesic arcs is a manifold of geodesic arcs equipped with a Poisson right M-algebra.*

Local expressions

Local coordinate expressions of Poisson manifolds of geodesic arcs can be found in the proof of Theorem 1. Throughout all the following text the standard summation convention with the summation from 1 to $\dim X$ is used.

Theorem 1. *Let $W(X)$ be a manifold of geodesic arcs of symmetric linear connection Γ on X. Then there exists a bijective correspondence between the set of all Poisson right M-algebras over $W(X)$ and the set of all ordered pairs (g, h) of tensor fields on X satisfying the following three conditions:*

a) *g is a tensor field of the type $(2, 1)$,*

b) *h is a tensor field of the type $(2, 0)$,*

c) *in each chart* $x \in \mathrm{At}(X)$ *the components* g^{ij}_k, h^{ij} *satisfy the relations:*

$$h^{ij} + h^{ji} = 0, \tag{3}$$

$$h^{il} h^{jk}_l + h^{jl} h^{ki}_l + h^{kl} h^{ij}_l = 0, \tag{4}$$

$$g^{il}_m g^{jk}_l - g^{jl}_m g^{ik}_l + g^{ik}_{ml} h^{lj} - g^{jk}_{ml} h^{li} = g^{lk}_m h^{ij}_l + R^k_{mlr} h^{li} h^{rj}, \tag{5}$$

$$g^{ij}_k - g^{ji}_k = h^{ij}_k, \tag{6}$$

$$g^{ij}_{kl} + g^{ij}_{lk} + g^{ji}_{kl} + g^{ji}_{lk} = \Big(R^i_{klm} + R^i_{lkm}\Big) h^{mj} + \Big(R^j_{klm} + R^j_{lkm}\Big) h^{mi}, \tag{7}$$

$$\begin{aligned}
&\tfrac{1}{2}\Big(h^{ij}_{lm} + h^{ij}_{ml} + R^i_{lmk} h^{kj} + R^i_{mlk} h^{kj} - R^j_{lmk} h^{ki} - R^j_{mlk} h^{ki}\Big)_r \\
&+\tfrac{1}{2}\Big(h^{ij}_{mr} + h^{ij}_{rm} + R^i_{mrk} h^{kj} + R^i_{rmk} h^{kj} - R^j_{mrk} h^{ki} - R^j_{rmk} h^{ki}\Big)_l \\
&+\tfrac{1}{2}\Big(h^{ij}_{rl} + h^{ij}_{lr} + R^i_{rlk} h^{kj} + R^i_{lrk} h^{kj} - R^j_{rlk} h^{ki} - R^j_{lrk} h^{ki}\Big)_m \\
&= \Big(R^i_{mrk} + R^i_{rmk}\Big) g^{kj}_l + \Big(R^i_{rlk} + R^i_{lrk}\Big) g^{kj}_m + \Big(R^i_{lmk} + R^i_{mlk}\Big) g^{kj}_r \\
&-\Big(R^j_{mrk} + R^j_{rmk}\Big) g^{ki}_l - \Big(R^j_{rlk} + R^j_{lrk}\Big) g^{ki}_m - \Big(R^j_{lmk} + R^j_{mlk}\Big) g^{ki}_r,
\end{aligned} \tag{8}$$

where R^i_{jkl} *are components of the Riemannian tensor field of the connection* Γ *and the lower indices which does not belong to the indexation of tensors* g, h *denote the corresponding covariant derivatives.*

Proof. For each x we consider a mapping

$$\psi : \operatorname{dom}\psi \ni \gamma \to \Big(x \circ \gamma(t), \frac{d}{dt}(x \circ \gamma(t))\Big)\Big|_{t=0} \in \operatorname{rng}\psi \subset \mathbb{R}^{2\dim X}, \tag{9}$$

where $\operatorname{dom}\psi = \{\gamma \in W(X) \mid \gamma(0) \in \operatorname{dom} x\}$. Since

$$\psi \circ \beta^{-1}\Big|_{\beta(\operatorname{dom}\psi)} \in \mathrm{At}(V),$$

the set of all charts of type (9) makes an atlas $\mathrm{At}(W(X))$. The coordinates of the chart $\psi \in \mathrm{At}(W(X))$ will be denoted by q^i, v^i.

Let us consider two tensor fields g and h on X satisfying the conditions a)–c). Let ϑ be the projection $\operatorname{dom}\psi \ni \gamma \to \gamma(0) \in \operatorname{dom} x$ and let Γ^i_{jk} be the components of the connection Γ. Expressions

$$\{q^i, q^j\} = h^{ij} \circ \vartheta, \tag{10}$$

$$\{q^i, v^j\} = \Big(\Big(g^{ij}_k - \Gamma^j_{kl} h^{il}\Big) \circ \vartheta\Big) v^k, \tag{11}$$

$$\begin{aligned}
\{v^i, v^j\} = \Big(\Big(&-\tfrac{1}{2} h^{ij}_{lm} - \tfrac{1}{2} R^i_{lmk} h^{kj} + \tfrac{1}{2} R^j_{lmk} h^{ki} \\
&- \Gamma^i_{lk} g^{kj}_m + \Gamma^j_{lk} g^{ki}_m + \Gamma^i_{kl} \Gamma^j_{rm} h^{kr}\Big) \circ \vartheta\Big) v^l v^m
\end{aligned} \tag{12}$$

are components of a skew-symmetric tensor field on $W(X)$. This fact makes one possible to define a skew-symmetric algebra structure $C^\infty(W(X)) \times C^\infty(W(X)) \ni (F,G) \to \{F,G\} \in C^\infty(W(X))$ by local coordinate expressions

$$\begin{aligned} \{F,G\}\Big|_{\operatorname{dom}\psi} &= \{q^i,q^j\}\frac{\partial F}{\partial q^i}\frac{\partial G}{\partial q^j} \\ &+ \{q^i,v^j\}\Big(\frac{\partial F}{\partial q^i}\frac{\partial G}{\partial v^j} - \frac{\partial F}{\partial v^j}\frac{\partial G}{\partial q^i}\Big) + \{v^i,v^j\}\frac{\partial F}{\partial v^i}\frac{\partial G}{\partial v^j}, \end{aligned} \tag{13}$$

where $\psi \in \mathrm{At}(W(X))$.

Using (6)–(8) we get

$$\partial_\xi\{F,G\} = \{\partial_\xi F,G\} + \{F,\partial_\xi G\}, \tag{14}$$

where a vector field ξ is defined on $W(X)$ by expressions

$$\xi\Big|_{\operatorname{dom}\psi} = v^i\frac{\partial}{\partial q^i} - \Big(\Gamma^i_{jk}\circ\vartheta\Big)v^jv^k\frac{\partial}{\partial v^i} \tag{15}$$

and ∂_ξ is the Lie derivative with respect to ξ.

Let us denote $\mathfrak{J}(F,G,H) = \{F,\{G,H\}\} + \{G,\{H,F\}\} + \{H,\{F,G\}\}$. From (3)–(5) we get $\mathfrak{J}(q^i,q^j,q^k) = \mathfrak{J}(q^i,q^j,v^k) = 0$ on $\operatorname{dom}\psi$. Consequently for all $H \in C^\infty(\operatorname{dom}\psi)$ it holds $\mathfrak{J}(q^i,q^j,H) = 0$. By the Lie differentiation with respect to ξ we obtain $\mathfrak{J}(v^i,q^j,H) + \mathfrak{J}(q^i,v^j,H) = 0$. Putting $H = v^k$ we can compute $\mathfrak{J}(q^i,v^j,v^k) = 0$. From the above relations for all $G,H \in C^\infty(\operatorname{dom}\psi)$ it holds $\mathfrak{J}(q^i,G,H) = 0$. By the next Lie derivative with respect to ξ we have $\mathfrak{J}(v^i,G,H) = 0$. For all $F,G,H \in C^\infty(\operatorname{dom}\psi)$ this implies $\mathfrak{J}(F,G,H) = 0$. Hence there exists a chart $\psi \in \mathrm{At}(W(X))$ at each arc of $W(X)$, the Jacobi identity is fulfilled on all $C^\infty(W(X))$ and so $C^\infty(W(X))$ is a Lie algebra. Besides, the Leibniz condition is fulfilled too and so $C^\infty(W(X))$ is a Poisson algebra over $W(X)$.

Let us consider a Poisson algebra over V such that β is a Poisson mapping. Then there exists a Poisson structure on TX such that the inclusion $V \subset TX$ and all the mappings $\lambda_p : TX \ni a \to pa \in TX$, where $p \in \mathbb{R}$, are Poisson mappings. Furthermore there exists a vector field η on TX such that $F \in C^\infty(TX)$, $p \in \mathbb{R}$ implies $\partial_\xi(F\circ\beta) = (\partial_\eta F)\circ\beta$, $\partial_\eta(F\circ\lambda_p) = (\partial_\eta F)\circ\lambda_p$. According to (14), for each $p \in \mathbb{R}$ the flow $\exp(p\eta)$ is a Poisson mapping. Thence for each $\mu \in M$ the mapping $R_\mu = \beta^{-1}\circ\lambda_{\mu(1)-\mu(0)}\circ\exp(\mu(0)\eta)\circ\beta$ is a Poisson mapping and $C^\infty(W(X))$ is a Poisson right M-algebra over $W(X)$.

Conversely, let $C^\infty(W(X))$ be a Poisson right M-algebra over $W(X)$. Then (13) holds (see, e.g., [1], [2]). If we sum up the derivatives of the relation $\{F\circ R_\mu, G\circ R_\mu\} = \{F,G\}\circ R_\mu$ with respect to $\mu(0)$ and with respect to $\mu(1)$ then putting $\mu = \mathrm{id}_{\langle 0,1\rangle}$ we see that (14) holds too.

Let $k \in \langle 0,1\rangle$, $\kappa : \langle 0,1\rangle \ni \tau \to k\tau \in \langle 0,1\rangle$. For Poisson mapping R_κ relations $q^i\circ R_\kappa = q^i$, $v^i\circ R_\kappa = kv^i$ hold. Thence

$$\begin{aligned} \frac{\partial^2\{q^i,v^j\}}{\partial v^r\,\partial v^s} &= \lim_{k\to 0}\frac{\partial^2\{q^i,v^j\}}{\partial v^r\,\partial v^s} = \lim_{k\to 0}\frac{1}{k}\frac{\partial^2\{q^i,kv^j\}}{\partial v^r\,\partial v^s} \\ &= \lim_{k\to 0}\frac{1}{k}\frac{\partial^2\{q^i\circ R_\kappa, v^j\circ R_\kappa\}}{\partial v^r\,\partial v^s} = \lim_{k\to 0}\frac{1}{k}\frac{\partial^2\{q^i,v^j\}\circ R_\kappa}{\partial v^r\,\partial v^s} \\ &= \lim_{k\to 0}k\frac{\partial^2\{q^i,v^j\}}{\partial v^r\,\partial v^s}\circ R_\kappa = 0. \end{aligned}$$

Similarly

$$\frac{\partial\{q^i,q^j\}}{\partial v^r} = \lim_{k\to 0}\frac{\partial\{q^i\circ R_\kappa, q^j\circ R_\kappa\}}{\partial v^r} = \lim_{k\to 0}\frac{\partial\{q^i,q^j\}\circ R_\kappa}{\partial v^r}$$
$$= \lim_{k\to 0} k\frac{\partial\{q^i,q^j\}}{\partial v^r}\circ R_\kappa = 0.$$

Hence further

$$\lim_{k\to 0}\{q^i,v^j\}\circ R_\kappa = \lim_{k\to 0}\{q^i,kv^j\} = \lim_{k\to 0} k\{q^i,v^j\} = 0,$$

we obtain (10) and $\{q^i,v^j\} = (\overline{g}^{ij}_k\circ\vartheta)v^k$ where $\overline{g}^{ij}_k$ are functions on $\operatorname{dom} x$. According to (13), the functions h^{ij} and $g^{ij}_k = \overline{g}^{ij}_k + \Gamma^j_{kl}h^{il}$ are components of tensor fields g and h on X such that (11) holds. The fields g and h satisfy conditions a) and b).

It remains to show that c) holds. From the first Lie derivative of (11) and the lowest three Lie derivatives of (10) with respect to ξ we obtain (7), (6), (12) and (8). Relation (3) is a consequence of (10). Relations (4) and (5) follow from Jacobi identities $\mathfrak{J}(q^i,q^j,q^k) = \mathfrak{J}(q^i,q^j,v^k) = 0$. This completes the proof.

Example. In-out manifolds defined in work (Klapka [6]) are isomorphic to Poisson manifolds of geodesic arcs whose Poisson submanifold of constant arcs is Abelian. In such a case relation (10) implies $h = 0$. Then by (5) and (6) the tensor field g is the field of structure tensors of fibration of cotangent commutative associative algebras over basis X.

The relation to classical Lagrangian mechanics

For a geometric formulation of the first order Lagrangian theory the formalism of fibred manifolds and theirs lower two prolongations is used (see, e.g., Krupka [7]). In case of mechanics we consider a fibred manifold $\pi : \mathbb{R}\times X \to \mathbb{R}$, where $\mathbb{R}$ is the manifold of real numbers and X is a smooth finite-dimensional manifold, with its prolongations $J^1(\mathbb{R}\times X)$, $J^2(\mathbb{R}\times X)$ and projections $\pi_1 : J^1(\mathbb{R}\times X)\to\mathbb{R}$, $\pi_{21} : J^2(\mathbb{R}\times X)\to J^1(\mathbb{R}\times X)$, $\pi_{20} : J^2(\mathbb{R}\times X)\to\mathbb{R}\times X$. We shall use the fiber charts with coordinates t, x^i, $\dot{x}^i$, $\ddot{x}^i$. A time evolution of mechanical system is described by local sections of the fibred manifold $\pi : \mathbb{R}\times X\to\mathbb{R}$ which are solutions of Euler–Lagrange equations

$$\epsilon_i\circ j^2\sigma = 0. \tag{16}$$

Here Euler–Lagrange expressions ϵ_i are components of π_{20}-horizontal 1-contact differential 2-form on $J^2(\mathbb{R}\times X)$ such that a Lagrange function L satisfying

$$\epsilon_i = \frac{d}{dt}\Big(\frac{\partial L}{\partial\dot{x}^i}\Big) - \frac{\partial L}{\partial x^i}, \tag{17}$$

where d/dt is the total derivative, locally exists. The Lagrange function is the component of the π_1-horizontal differential 1-form on $J^1(\mathbb{R}\times X)$. Note that by the usual notation practice in the classical mechanics all necessary projections are omitted (for instance, the restriction of projection π_{21} absents in the second term of the right-hand side of (17)). We suppose that the Euler–Lagrange expressions are regular, that is, for each point of $J^2(\mathbb{R} \times X)$

$$\det\Big(\frac{\partial \epsilon_k}{\partial \ddot{x}^l}\Big) \neq 0. \tag{18}$$

Let us consider a set $E(\mathbb{R} \times X)$ of all solutions $\sigma_{ab} : \langle a, a+b\rangle \to \mathbb{R} \times X$ of equations (16), where $a, b \in \mathbb{R}$, $b > 0$. Then there exists a bijection

$$\chi : E(\mathbb{R} \times X) \ni \sigma_{ab} \to (b, j^1\sigma_{ab}(a)) \in U, \tag{19}$$

where U is an open subset of the manifold $(0, \infty) \times J^1(\mathbb{R} \times X)$. The Poisson structure of this manifold can be found by the methods of classical mechanics. From classical canonical relations

$$\{x^i, x^j\} = 0,\ \Big\{x^i, \frac{\partial L}{\partial \dot{x}^j}\Big\} = \delta^i_j,\ \Big\{\frac{\partial L}{\partial \dot{x}^i}, \frac{\partial L}{\partial \dot{x}^j}\Big\} = 0$$

we obtain

$$\begin{gathered}\{x^k, x^l\} = 0,\\ \{x^k, \dot{x}^m\}\frac{\partial \epsilon_l}{\partial \ddot{x}^m} = \delta^k_l,\\ \{\dot{x}^k, \dot{x}^l\}\frac{\partial \epsilon_r}{\partial \ddot{x}^k}\frac{\partial \epsilon_s}{\partial \ddot{x}^l} = \frac{1}{2}\Big(\frac{\partial \epsilon_s}{\partial \dot{x}^r} - \frac{\partial \epsilon_r}{\partial \dot{x}^s}\Big).\end{gathered} \tag{20}$$

Note that in the classical mechanics the coordinates on $\mathbb{R}$ and $(0, \infty)$ are considered as Casimir functions. The set $E(\mathbb{R} \times X)$ equipped with a Poisson algebra such that χ is an isomorphism of Poisson manifolds will be called a *Poisson manifold of solutions of Euler–Lagrange equations*. The following theorem deals with the relation between Poisson manifolds of solutions of Euler–Lagrange equations and Poisson manifolds of geodesic arcs.

Theorem 2. *Let $L_\pi : W(\mathbb{R} \times X) \to W(\mathbb{R})$ be a fibred Poisson manifold of geodesic arcs such that all fibers are Poisson submanifolds, fibers over constant arcs are Abelian and fibers over non-constant arcs are symplectic. Then $W(\mathbb{R} \times X)$ is a union of three disjoint Poisson submanifolds such that the two of them are isomorphic to certain Poisson manifold of solutions of Euler–Lagrange equations and the third of them is an Abelian fibration of manifolds of geodesic arcs of Levi–Civita connection on X.*

Proof. Let us consider a chart $x \in \mathrm{At}(\mathbb{R} \times X)$ such that x^0 is a coordinate on $\mathbb{R}$, x^1, x^2, …, $x^{\dim X}$ are coordinates on X and $\operatorname{rng} x$ is an open ball in $\mathbb{R}^{\dim X + 1}$. Since $W(\mathbb{R} \times X)$ is a fibred manifold of geodesic arcs over basis $W(\mathbb{R})$, we get $R^i_{jkl} = 0$ for $i = 0$. Hence, the coordinate x^0 may be chosen in such a way that for $i = 0$

$$\Gamma^i_{jk} = 0. \tag{21}$$

Since the fibers over constant arcs are Abelian, by (10) and (11) we obtain $h^{ij} = 0$, $g^{ij}_k = 0$, where $k \neq 0$. Since the fibers over non-constant arcs are Poisson submanifolds, by (11) we obtain $g^{i0}_0 = g^{0i}_0 = 0$.

Throughout all the following text we will not consider the zero value of indices i, j, k, l, m. We denote $g^{ij}_0 = g^{ij}$ and $x^0 = t$. Taking into account that the fibers over non-constant arcs are symplectic we obtain

$$\det(g^{ij}) \neq 0 \tag{22}$$

at each point of X. By (6) $g^{ij} = g^{ji}$ follows. Relations (3) - (5) are fulfilled identically. Relation (7) can be expressed in the form

$$\frac{\partial g^{ij}}{\partial x^k} + \Gamma^i_{km} g^{mj} + \Gamma^j_{km} g^{im} = 0, \ \frac{\partial g^{ij}}{\partial t} + \Gamma^i_{0m} g^{mj} + \Gamma^j_{0m} g^{im} = 0. \tag{23}$$

By a straightforward computation we get

$$\Gamma^i_{jk} = \tfrac{1}{2} g^{il} \Big(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \Big), \tag{24}$$

$$\Gamma^i_{j0} = \Gamma^i_{0j} = \tfrac{1}{2} g^{il} \Big(\frac{\partial g_{lj}}{\partial t} + \Phi_{lj} \Big), \tag{25}$$

where g_{ij} are solutions of equations $g_{ij} g^{jk} = \delta^k_i$ and Φ_{ij} are functions on $\operatorname{dom} x$ such that $\Phi_{ij} + \Phi_{ji} = 0$. Relation (8) can be written according to (24) and (25) in the form

$$\frac{\partial \Phi_{ij}}{\partial x^k} + \frac{\partial \Phi_{jk}}{\partial x^i} + \frac{\partial \Phi_{ki}}{\partial x^j} = 0, \ \frac{\partial \Phi_{ij}}{\partial t} + \frac{\partial F_j}{\partial x^i} - \frac{\partial F_i}{\partial x^j} = 0, \tag{26}$$

where $F_i = g_{ij} \Gamma^j_{00}$. Thence the Poincaré lemma enables us to show that there exist functions A_i, φ on $\operatorname{dom} x$ satisfying

$$\Phi_{ij} = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}, \ F_i = \frac{\partial A_i}{\partial t} - \frac{\partial \varphi}{\partial x^i}. \tag{27}$$

Since the Lagrange function

$$L = \tfrac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + A_i \dot{x}^i + \varphi \tag{28}$$

exists on $J^1(\operatorname{dom} x)$, the components

$$\epsilon_i = g_{ij} \Big(\ddot{x}^j + \Gamma^j_{kl} \dot{x}^k \dot{x}^l + 2\Gamma^j_{k0} \dot{x}^k + \Gamma^j_{00} \Big) \tag{29}$$

of a π_{20}-horizontal 1-contact differential 2-form on $J^2(\mathbb{R} \times X)$ are Euler–Lagrange expressions. According to (18) and (22) they are regular. Let us denote the corresponding manifold of solutions of Euler–Lagrange equations by $E(\mathbb{R} \times X)$.

In view of (21) $W(\mathbb{R}) = W_1(\mathbb{R}) \cup W_2(\mathbb{R}) \cup W_3(\mathbb{R})$, where $W_1(\mathbb{R})$ is the open submanifold of all increasing geodesic arcs, $W_2(\mathbb{R})$ is the open submanifold of all decreasing geodesic arcs and $W_3(\mathbb{R})$ is the closed submanifold of all constant geodesic arcs. Let us decompose the manifold $W(\mathbb{R} \times X)$ into the three disjoint submanifolds $W_1(\mathbb{R} \times X) = L_\pi^{-1}(W_1(\mathbb{R}))$, $W_2(\mathbb{R} \times X) = L_\pi^{-1}(W_2(\mathbb{R}))$ and $W_3(\mathbb{R} \times X) = L_\pi^{-1}(W_3(\mathbb{R}))$. These submanifolds are Poisson ones because of $W_1(\mathbb{R} \times X)$, $W_2(\mathbb{R} \times X)$ are open ones and $W_3(\mathbb{R} \times X)$ is by assumption the union of Abelian Poisson manifolds over constant geodesic arcs. Clearly, the submanifold $W_3(\mathbb{R} \times X)$ is Abelian too.

Let us suppose $\gamma \in W_1(\mathbb{R} \times X)$. Then there exists a unique mapping σ such that $\operatorname{dom} \sigma = \operatorname{rng}(\pi \circ \gamma)$, $\gamma = \sigma \circ \pi \circ \gamma$. It holds $\sigma \in E(\mathbb{R} \times X)$. The induced mapping $W_1(\mathbb{R} \times X) \to E(\mathbb{R} \times X)$ is an isomorphism of Poisson manifolds.

Let us suppose $\gamma \in W_2(\mathbb{R} \times X)$ and define the mapping $\mu : \langle 0, 1\rangle \ni \tau \to (1-\tau) \in \langle 0, 1\rangle$. In such a case $\gamma \circ \mu = R_\mu(\gamma) \in W(\mathbb{R} \times X)$. Since $\pi \circ \gamma$ and μ are decreasing mappings, $\pi \circ \gamma \circ \mu$ is an increasing mapping and so $\gamma \circ \mu \in W_1(\mathbb{R} \times X)$. Since R_μ is a Poisson mapping and since $R_\mu^{-1} = R_\mu$ holds, the mapping $W_2(\mathbb{R} \times X) \ni \gamma \to \gamma \circ \mu \in W_1(\mathbb{R} \times X)$ is an isomorphism of Poisson manifolds. Consequently, the fact that there is an isomorphism between $W_1(\mathbb{R} \times X)$ and $E(\mathbb{R} \times X)$ implies the existence of isomorphism between $W_2(\mathbb{R} \times X)$ and $E(\mathbb{R} \times X)$.

For each $a \in \mathbb{R}$ the set $\pi^{-1}(a)$ is a geodesic submanifold of the manifold $\mathbb{R} \times X$ that is diffeomorphic to X. According to (24) the connection on $\pi^{-1}(a)$ is the Levi–Civita one. Let $W(\pi^{-1}(a))$ be the manifold of geodesic arcs of this connection. Taking into account

$$W_3(\mathbb{R} \times X) = \bigcup_{a \in \mathbb{R}} W\Big(\pi^{-1}(a)\Big), \tag{30}$$

we see that $W_3(\mathbb{R} \times X)$ is the Abelian fibration of manifolds of geodesic arcs of Levi–Civita connection on X over basis $\mathbb{R}$. This completes the proof.

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