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ALMOST PRODUCT STRUCTURES IN MECHANICS

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ABSTRACT. There are two different meanings for constrained systems in Lagrangian mechanics: "internal" constraints imposed by the singularity of the Lagrangian function, or "external"constraints imposed by forces of constraint acting on the system. In the first case, it is usual to apply the Gotay-Nester algorithm in the framework of presymplectic geometry and, in the second case, the classical technique of Lagrange multipliers. The aim of this paper it is to give a new look at singular and non-holonomic Lagrangian systems in the framework of almost product structures.

1. INTRODUCTION

We use almost product structures to study, at a first glance, different dynamical systems: degenerate and non-holonomic Lagrangian systems. Roughly speaking, an almost product structure on a manifold M consists of two complementary distributions on M. Therefore if $L : TQ \longrightarrow \mathbb{R}$ is a singular Lagrangian function, it is very natural to take an almost product structure on TQ such that one of the two complementary distributions is just the singular distribution $\ker \omega_L$. The dynamics may be obtained by projecting the system on the complementary distribution. The use of almost product structures in order to obtain the "true" dynamics of a singular system was proposed in several recent papers (see de León, Martín de Diego & Pitanga [8], de León & Martín de Diego [11], Pitanga [15], Dubrovin, Giordano, Marmo & Simone [5] and references therein).

Given a Lagrangian system with non-holonomic constraints we construct an almost product structure on the tangent bundle of the configuration manifold such that the projection of the Euler-Lagrange vector field for the free Lagrangian system gives the dynamics of the system. We go further in the direction of the geometric formulation of non-holonomic Lagrangian systems started by Cariñena & Rañada [3], Rañada [16], Sarlet, Cantrijn & Saunders [17], de León & Martín de Diego[9].

By using both constructions, we study the general case of degenerate Lagrangian systems with non-holonomic constraints. In this general setting, several almost product structures arise.

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2. Almost product structures

An almost product structure on a manifold M is a tensor field F of type $(1,1)$ on M such that $F^2 = id$ (see [13]). If we set

$$\mathcal{A} = \frac{1}{2}(id + F) \quad , \quad \mathcal{B} = \frac{1}{2}(id - F) \,,$$

then $\mathcal{A}$ and $\mathcal{B}$ are complementary projectors. We denote by $Im\mathcal{A}$ and $Im\mathcal{B}$ the corresponding complementary distributions. We also denote by $\mathcal{A}^*$ and $\mathcal{B}^*$ the transpose operators and $Im\mathcal{A}^*$ and $Im\mathcal{B}^*$ will be their corresponding images.

Let (M, ω) be a presymplectic manifold with a presymplectic form ω. An almost product structure F on M is said to be adapted to ω if $\ker\omega = \ker\mathcal{A}$. Define the linear mapping $\flat : \mathfrak{X}(M) \longrightarrow \wedge^1(M)$ by $\flat(X) = i_X\omega$. If F is adapted to ω, the restriction of $\flat$ to the distribution $\mathcal{A}$ induces an isomorphism of C^∞-modules $\flat : Im\mathcal{A} \longrightarrow Im\mathcal{A}^*$. Then, for a function f on M, there exists a unique vector field $X_{f,\mathcal{A}} \in Im\mathcal{A}$ such that

$$i_{X_{f,\mathcal{A}}}\omega = \mathcal{A}^*(df) \,. \tag{1}$$

A bracket of functions can be defined as follows:

$$\{f, g\}_\mathcal{A} = \omega(X_{f,\mathcal{A}}, X_{g,\mathcal{A}}) \,, \; f, g \in C^\infty(M) \,.$$

$\{\ ,\ \}_\mathcal{A}$ satisfies all the properties of a Poisson bracket except the Jacobi identity. The Jacobi identity is equivalent to the integrability of the almost product structure F (see Dubrovin *et al* [5]). Let us recall that an almost product structure F is said to be integrable if both distributions $Im\mathcal{A}$ and $Im\mathcal{B}$ are integrable. In our case, the distribution $Im\mathcal{B} = \ker\omega$ is always integrable, but $Im\mathcal{A}$ is not necessarily so.

3. Lagrangian systems

Let Q be an n-dimensional manifold, TQ its tangent bundle, and $\tau_Q : TQ \longrightarrow Q$ the canonical projection. Denote by $\{q^A, \dot{q}^A; 1 \leq A \leq n\}$ the fibered coordinates on TQ. Let J be the canonical almost tangent structure and C the Liouville vector field on TQ.

Consider a Lagrangian function L on TQ. Let $E_L = CL - L$ be the energy associated with L. We denote by $\alpha_L = J^*(dL)$ the Poincaré-Cartan 1-form and, by $\omega_L = -d\alpha_L$ the Poincaré-Cartan 2-form. A Lagrangian L is said to be regular if its Hessian matrix

$$(W_{AB}) = \left(\frac{\partial^2 L}{\partial\dot{q}^A\partial\dot{q}^B}\right) \tag{2}$$

is regular. If L is regular, ω_L is symplectic and, in this case, the motion equation

$$i_X\omega_L = dE_L \tag{3}$$

has a unique solution ξ_L (the Euler-Lagrange vector field). Moreover, ξ_L turns out to be a second order differential equation (hereafter shortened as SODE), that is, ξ_L verifies that $J\xi_L = C$. The induced Poisson bracket on $C^\infty(TQ)$ is defined by

$$\{f, g\}_L = \omega_L(X_f, X_g) \ , \ \forall f, g \in C^\infty(TQ) \ ,$$

where X_f denotes the Hamiltonian vector field with Hamiltonian energy f, that is, $i_{X_f}\omega_L = df$. Hence $\xi_L = X_{E_L}$ and $\{f, E_L\}_L = \xi_L(f)$.

If $L : TQ \longrightarrow \mathbb{R}$ is singular, i.e., the Hessian matrix (2) is not regular, Equation (3) has no solution, in general, and even if it exists it will not be unique nor a SODE. We suppose, for the sake of simplicity, that the system admits a global dynamics, that is, a solution ξ on TQ of Equation (3) exists. Then, if $Z \in \ker \omega_L$, $\xi + Z$ is also a solution of Equation (3).

Let T^*Q be the cotangent bundle of Q with canonical projection $\pi : T^*Q \longrightarrow Q$. We will denote by $Leg : TQ \longrightarrow T^*Q$ the Legendre map defined by L

$$Leg : (q^A, \dot{q}^A) \longrightarrow (q^A, \tilde{p}_A) \ ,$$

with $\tilde{p}_A = \partial L / \partial \dot{q}^A$.

L is regular iff Leg is a local diffeomorphism. For the sake of simplicity, we suppose that L is hyper-regular, that is, Leg is a global diffeomorphism. If λ_Q is the Liouville 1-form and $\omega_Q = -d\lambda_Q$ is the canonical symplectic form on T^*Q, we have that

$$Leg^*\omega_Q = \omega_L \ , \ Leg^*\lambda_Q = \alpha_L \ .$$

We define the Hamiltonian function H on T^*Q by $H = E_L \circ Leg^{-1}$. If X_H denotes the Hamiltonian vector field of H we know that ξ_L and X_H are Leg-related. Then, both dynamics are equivalent.

Suppose that L is singular but almost regular, i.e., $M_1 = Leg(TQ)$ is a submanifold of T^*Q and Leg is a submersion onto M_1 with connected fibers. We denote by $Leg_1 : TQ \longrightarrow M_1$ the restriction of $Leg : TQ \longrightarrow T^*Q$ to its image. The submanifold M_1 will be called the primary constraint submanifold. The energy E_L is constant along the fibers of Leg and, therefore, E_L projects onto a function h_1 on M_1, i.e., $h_1(Leg(x)) = E_L(x)$, $\forall x \in TQ$. Let $\{F, G\} = \omega_Q(X_F, X_G)$, $\forall F, G \in C^\infty(T^*Q)$ be the Poisson bracket on T^*Q induced by ω_Q. If we denote by $i : M_1 \longrightarrow T^*Q$ the natural embedding of M_1 into T^*Q, then we obtain a presymplectic system (M_1, ω_1, dh_1), where $\omega_1 = i^*\omega_Q$.

4. Singular Lagrangian systems and almost product structures

Suppose that (TQ, ω_L, dE_L) admits a global dynamics and take an almost product structure $(\mathcal{A}, \mathcal{B})$ adapted to ω_L. Then, we can fix a unique solution of the dynamics by taking the vector field ξ given by

$$i_\xi \omega_L = dE_L \quad and \quad \xi \in Im\mathcal{A} \ .$$

The presymplectic system (M_1, ω_1, dh_1) also admits a global dynamics ([6]), that is, there exists a vector field X on M_1 such that

$$i_X\omega_1 = dh_1, \tag{4}$$

and each vector field $X+Z$ with $Z \in \ker \omega_1$ is also a solution of this equation. If $(\mathcal{A}, \mathcal{B})$ is projectable onto an almost product structure on M_1 then ξ is projectable onto a vector field X which is a solution of Equation (4).

Since M_1 is a submanifold of T^*Q, there appear $n-k$ independent constraints, the primary constraints, ϕ^α which describe M_1 (see [4]). If H is an arbitrary extension of h_1 to T^*Q, all the Hamiltonian functions of the form

$$\tilde{H} = H + \lambda_\alpha \phi^\alpha \,, \tag{5}$$

where λ_α are Lagrange multipliers, are weakly equal, that is, $\tilde{H}_{/M_1} = H_{/M_1} = h_1$. This shows that there exists an ambiguity in the description of the dynamics. Since ω_Q is symplectic, a solution of the equation $i_X \omega_Q = dH$ always exists. The constraints must be preserved in the time or, equivalently, the solution X must be tangent to M_1. Then, we get

$$\left(\{\phi^\beta, \tilde{H}\} + \lambda_\alpha \{\phi^\beta, \phi^\alpha\} \right)_{/M_1} = 0 \,. \tag{6}$$

The vanishing of the expressions (6) implies that some of the arbitrary functions λ_α may be determined and the rest remains completely undetermined.

Dirac gave a classification of the constraints generated by this algorithm in order to clarify the ambiguity of the dynamics ([4, 2, 18]). A constraint ϕ of M_1 is said to be first class if $\{\phi, \phi^\alpha\}_{/M_1} = 0$ for each constraint ϕ^α of M_1, and second class otherwise. Then, the coefficients of the primary first class constraints on M_1 in (5) are completely undetermined, while the coefficients of the primary second class constraints are completely fixed.

We will distinguish three particular cases: all the primary constraints are second class, all them are first class and there exist first and second class constraints.

4.1. All the primary constraints are second class

Denote by Φ^α, $1 \leq \alpha \leq s$, the constraints of M_1. The matrix with entries $C^{\alpha\beta} = \{\Phi^\alpha, \Phi^\beta\}$ is non-singular on M_1 and, in the sequel, we assume for simplicity that this matrix is non-singular in the entire phase space T^*Q. This matrix is skewsymmetric and, then, the number of second class constraints is even. Denote by $(C_{\alpha\beta})$ its inverse matrix.

Let D be the smooth distribution generated by the vector fields X_{Φ^α}. A direct computation shows that

$$D^\perp(x) = \{v \in T_xT^*Q \;/\; \omega_Q(x)(v,w) = 0 \,,\ \forall w \in D(x)\} = T_xM_1 \,, \quad \forall x \in M_1 \,.$$

Let $\widetilde{\mathcal{B}} : D \oplus D^\perp \longrightarrow D$ be the projection onto D along $D^\perp$ and $\widetilde{\mathcal{A}} = id - \widetilde{\mathcal{B}}$. The projector $\widetilde{\mathcal{B}}$ is given by

$$\widetilde{\mathcal{B}} = C_{\alpha\beta} X_{\Phi^\alpha} \otimes d\Phi^\beta \,.$$

Take the 2-form $\Omega_D = \widetilde{\mathcal{A}}^* \omega_Q$ (that is, $\widetilde{\mathcal{A}}^*\omega_Q(X,Y) = \omega_Q(\widetilde{\mathcal{A}}X, \widetilde{\mathcal{A}}Y)$). Ω_D is an almost presymplectic form with constant rank $2n-s$ and, moreover, the almost product structure $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ is adapted to Ω_D. Notice that, if $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ is integrable, then

Ω_D is closed and, hence, presymplectic. Thus, we can define a bracket $\{F, G\}_D$, called Dirac bracket, on T^*Q as follows:

$$\{F, G\}_D = \Omega_D(X_F, X_G) = \{F, G\} - \{F, \Phi^\alpha\} C_{\alpha\beta} \{\Phi^\beta, G\} .$$

Consider now the projected Hamiltonian function $h_1 : M_1 \longrightarrow \mathbb{R}$. We can extend h_1 to a function H on T^*Q and consider the Hamiltonian functions of the form:

$$\tilde{H} = H + \lambda_\alpha \Phi^\alpha .$$

Take the Hamiltonian vector field $X_{\tilde{H}}$. The consistency of the theory demands that the constraints Φ_α be preserved by $X_{\tilde{H}}$; geometrically this means that the vector field $X_{\tilde{H}}$ must be tangent to M_1.

The vector field

$$\widetilde{\mathcal{A}} X_H = X_H - C_{\alpha\beta} \{\Phi^\beta, H\} X_{\Phi^\alpha} .$$

is tangent to M_1 and its restriction $(\widetilde{\mathcal{A}} X_H)_{/M_1}$ is the unique solution of the motion equations, that is,

$$i_{(\widetilde{\mathcal{A}} X_H)_{/M_1}} \omega_1 = dh_1 ,$$

because ω_1 is symplectic.

4.2. All the primary constraints are first class

We denote by ϕ^i, $1 \leq i \leq p$, the first class constraints. Since $\{\phi^i, \phi^j\}_{/M_1} = 0$, then X_{ϕ_i}, $1 \leq i \leq p$, the Hamiltonian vector field of ϕ^i, is tangent to M_1. Notice that the submanifold M_1 is coisotropic into T^*Q.

Since $\ker \omega_1$ is generated by the restrictions of the Hamiltonian vector fields X_{ϕ^i}, in order to fix the gauge, we take an almost product structure $(\mathcal{A}_1, \mathcal{B}_1)$ on M_1 adapted to $\ker \omega_1$. Moreover, if the almost product structure is integrable, we can define a Poisson bracket on M_1 as follows:

$$\{f, g\}_{\mathcal{A}_1} = \omega_1(X_{f,\mathcal{A}_1}, X_{g,\mathcal{A}_1}) , \; \forall f, g \in C^\infty(M_1) ,$$

where $X_{f,\mathcal{A}_1}$ and $X_{g,\mathcal{A}_1}$ are the unique vector fields on M_1 which belong to $Im\mathcal{A}_1$ and verify that $i_{X_{f,\mathcal{A}_1}} \omega_1 = \mathcal{A}_1^* df$ and $i_{X_{g,\mathcal{A}_1}} \omega_1 = \mathcal{A}_1^* dg$, respectively. Therefore, if ξ is a solution of the motion equation, that is, $i_\xi \omega_1 = dh_1$, we can select a unique solution $\mathcal{A}_1 \xi$ such that $\mathcal{A}_1 \xi \in Im\mathcal{A}_1$. Thus, we have fixed the gauge.

Now, consider an arbitrary extension H to T^*Q of the Hamiltonian $h_1 : M_1 \longrightarrow \mathbb{R}$. Since we have a global dynamics, the Hamiltonian vector field X_H is tangent to M_1, i.e., $(X_H)_{/M_1} \in \mathfrak{X}(M_1)$ and

$$i_{(X_H)_{/M_1}} \omega_1 = dh_1 .$$

We fix the gauge by taking $\mathcal{A}_1((X_H)_{/M_1})$.

4.3. There exist first and second class constraints

We denote by Φ^α, $1 \leq \alpha \leq s$, the second class constraints and by ϕ^i, $1 \leq i \leq p$, the first class constraints. As in the first case, we can construct an almost product structure $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ on T^*Q with $\widetilde{\mathcal{B}}$ given by

$$\widetilde{\mathcal{B}} = C_{\alpha\beta} X_{\Phi^\alpha} \otimes d\Phi^\beta \ .$$

Here, $(C_{\alpha\beta})$ is the inverse matrix of $(\{\Phi^\alpha, \Phi^\beta\})$. We also have the almost presymplectic form $\Omega_D = \widetilde{\mathcal{A}}^*\omega_Q$ with constant rank $2n - s$ and the Dirac bracket

$$\{F, G\}_D = \{F, G\} - \{F, \Phi^\alpha\} C_{\alpha\beta} \{\Phi^\beta, G\} \ .$$

If we consider an arbitrary extension H of the Hamiltonian h_1, since we have a global dynamics, then the vector field $\widetilde{\mathcal{A}}(X_H)$ is tangent to M_1. Now, we fix the gauge taking the vector field $\mathcal{A}_1(\widetilde{\mathcal{A}}(X_H)_{/M_1})$, where $(\mathcal{A}_1, \mathcal{B}_1)$ is some almost product structure adapted to ω_1.

Remark 4.1. For an arbitrary Lagrangian system (TQ, ω_L, dE_L), we must apply the Gotay-Nester algorithm in order to obtain a final submanifold where there exists consistent solutions of the dynamics. From the equivalence theorem ([7]), this is equivalent to apply the algorithm to the presymplectic system (M_1, ω_1, dh_1) and we also have a final constraint submanifold M_f of M_1. In a similar way, we can obtain a classification of the constraints of M_f and then construct suitable almost product structures that guarantees the consistency and unicity of the dynamics.

Remark 4.2. If the almost product structure $(\mathcal{A}, \mathcal{B})$ on TQ is projectable on TQ and ξ is not a SODE then it is possible to construct from the projection $\tilde{\xi}$ a submanifold S of TQ where there exists a solution ξ_S of the dynamics which verifies the SODE condition, that is, $J(\xi_S)(x) = C(x)$, $\forall x \in S$. The submanifold S is diffeomorphic to M_1 and the dynamics on both sides are equivalent.

5. Non-holonomic regular Lagrangian systems and almost product structures

We assume that a regular lagrangian L is subjected to a system of m independent non-holonomic constraints $\{\phi_i; 1 \leq i \leq m,\ m \leq n\}$ which are affine in the velocities. This means that

$$\phi_i = \hat{\mu}_i + h_i^V \ ,$$

where $\mu_i = (\mu_i)_A(q)dq^A$ is a 1-form on Q and $h_i^V = h_i \circ \tau_Q$ with $h_i \in C^\infty(Q)$. Here $\hat{\mu} : TQ \longrightarrow \mathbb{R}$ denotes the function defined by

$$\hat{\mu}(X_q) = \langle \mu(q), X_q \rangle \ , \forall X_q \in T_qQ \ .$$

We have to restrict the dynamics to the submanifold $\bar{P}_1$ of TQ defined by the vanishing of the functions ϕ_i. The dynamics of the constrained system is obtained solving the following system of equations:

$$\left\{ \begin{array}{rcl} i_X \omega_L & = & dE_L + \lambda^i \mu_i^V \ , \\ d\phi_i(X) & = & 0 \ , \end{array} \right. \tag{7}$$

where $\mu_i^V = \tau_Q^* \mu_i$. The functions λ^i are Lagrange multipliers.

For any m-tuple $\lambda = (\lambda^1, \ldots, \lambda^m)$, we have the vector field Y_λ given by

$$Y_\lambda = \xi_L + \lambda^i Z_i \ ,$$

where Z_i are the vertical vector fields (see [3]) defined by

$$i_{Z_i} \omega_L = \mu_i^V \ .$$

Notice that Y_λ verifies the first equation of the system (7). If we demand that Y_λ satisfies the second condition, we obtain that

$$0 = \left(\{\phi_j, E_L\}_L + \lambda^i Z_i(\phi_j) \right)_{/\bar{P}_1} \ .$$

Denote by C the matrix on TQ of order m whose entries are $C_{ij} = Z_i(\phi_j)$. If C is regular, then the Lagrange multipliers λ^i are uniquely determined on TQ. In fact, assume that the matrix C is regular on TQ. Consider the (1,1) tensor field $\mathcal{Q}$ given by

$$\mathcal{Q} = C^{ij} Z_j \otimes d\phi_i \ ,$$

where C^{ij} are the entries of the inverse matrix of C, that is, $C^{ij} C_{jk} = \delta^i_k$. A direct computation shows that $\mathcal{Q}^2 = \mathcal{Q}$. If we set $\mathcal{P} = id - \mathcal{Q}$, then $(\mathcal{P}, \mathcal{Q})$ is an almost product structure on the phase space TQ. Moreover, we have that the vector field $((\mathcal{P}(\xi_L))_{/\bar{P}_1}$ is the unique solution of equations (7). In fact,

$$\mathcal{P}(\xi_L) = \xi_L - C^{ij} \xi_L(\phi_i) Z_j \ ,$$

which implies that $(\mathcal{P}(\xi_L))_{/\bar{P}_1}$ verifies the equation (7) for $\lambda^i = -\left(C^{ji}\xi_L(\phi_j)\right)_{/\bar{P}_1}$.

Remark 5.1. Suppose that a Lagrangian system is subjected to holonomic constraints $h_i \in C^\infty(Q)$, $(1 \le i \le m)$, with $m < n$. That is, the velocities do not enter into the constraint equations. In geometrical terms the motion equations are

$$(8) \qquad \left\{ \begin{array}{rcl} i_X \omega_L & = & dE_L + \lambda^i dh_i^V \ , \\ dh_i^c(X) & = & 0 \, , \\ dh_i^V(X) & = & 0 \, . \end{array} \right.$$

Since $dh_i^V(X) = h_i^c$ because X is a SODE, we can first study the system as a Lagrangian system subjected to the non-holonomic constrains h_i^c and solve the motion equations

$$(9) \qquad \left\{ \begin{array}{rcl} i_X \omega_L & = & dE_L + \lambda^i dh_i^V \ , \\ dh_i^c(X) & = & 0 \, , \end{array} \right.$$

and later, we impose the constraints $h_i^V = 0$. In this manner, a holonomic system is, in some sense, a special case of a non-holonomic system.

If we suppose that the Lagrangian is hyper-regular, that is the Legendre map is a diffeomorphism, the study of the Hamiltonian formalism is straightforward, because we can transport all the above constructions to the Hamiltonian side.

6. Non-holonomic degenerate Lagrangian systems

Consider now a singular Lagrangian system subjected to a set of non-holonomic constraints $\{\phi_i\}, (1 \leq i \leq m)$ as in the previous section.

Let (TQ, ω_L, dE_L) be a Lagrangian system which admits a global dynamics, i.e., $\ker \omega_L(E_L) = 0$. Assume that the constraints are Leg-projectable, i.e., $V(\ker \omega_L)(\phi_i) = 0$. As a consequence, we obtain that

$$\mu_i^V(\ker \omega_L) = 0 \ , \ (1 \leq i \leq m) \ .$$

We now suppose that there exists an almost product structure $(\mathcal{A}, \mathcal{B})$ adapted to the presymplectic form ω_L ($\ker \omega_L = \ker \mathcal{A}$). Then, we can fix a unique solution ξ of the free Lagrangian system as follows:

$$i_\xi \omega_L = dE_L \ and \ \xi \in Im\mathcal{A} \ .$$

Also, for each i, $1 \leq i \leq m$, we have a unique vector field Z_i such that

$$i_{Z_i}\omega_L = \mu_i^V \ and \ Z_i \in Im\mathcal{A} \ .$$

As in Section 5, if the matrix $C = (C_{ij}) = (Z_i(\phi_j))$, $(1 \leq i, j \leq m)$ is regular we can construct an almost product structure $(\mathcal{P}, \mathcal{Q})$ on TQ as follows:

$$\mathcal{Q} = C^{ij} Z_j \otimes d\phi_i \ and \ \mathcal{P} = id - \mathcal{Q} \ .$$

Let $\bar{P}_1$ be the submanifold of TQ defined by the vanishing of all the constraints. We deduce that $(\mathcal{P}(\xi))_{/\bar{P}_1}$ is a solution of the dynamics of the constrained system.

In order to obtain the Hamiltonian formalism, consider the presymplectic system (M_1, ω_1, dh_1) subjected to the constraints $\bar{\phi}_i$, $(1 \leq i \leq m)$ where $\bar{\phi}_i$ are defined by $\bar{\phi}_i(Leg(x)) = \phi_i(x), \forall\, x \in TQ$. Denote by $\bar{M}_1$ the submanifold of M_1 defined by the vanishing of all the constraints $\bar{\phi}_i$. It is clear that $Leg(\bar{P}_1) = \bar{M}_1$.

If the almost product structure $(\mathcal{A}, \mathcal{B})$ is projectable onto M_1 (see [8]) we obtain an almost product structure $(\mathcal{A}_1, \mathcal{B}_1)$ on M_1 adapted to ω_1. Moreover, the vector field $\xi \in Im\mathcal{A}$ solution of the motion equation is projectable on M_1 onto a vector field $\tilde{\xi}$ solution of the equation

$$i_{\tilde{\xi}}\omega_1 = dh_1 \ .$$

From construction the almost product structure $(\mathcal{P}, \mathcal{Q})$ is also projectable onto an almost product structure $(\bar{\mathcal{P}}, \bar{\mathcal{Q}})$ on M_1.

As in Section 4 we can classify the constraints of M_1 in first class and second class constraints. We consider three different cases:

6.1. All the primary constraints are second class

As in Section 4.1, we can construct an almost product structure $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ on T^*Q. Moreover, we have the almost product structure $(\bar{\mathcal{P}}, \bar{\mathcal{Q}})$ on M_1. Take now an arbitrary extension H of the Hamiltonian h_1 on M_1. We fix the unique solution (because $(\bar{M}_1, \omega_1)$ is symplectic) of the constrained dynamics by taking the vector field on $\bar{M}_1$

$$\left(\bar{\mathcal{P}}(\widetilde{\mathcal{A}}(X_H)_{/M_1})\right)_{/\bar{M}_1} .$$

In fact, $\widetilde{\mathcal{A}}(X_H)_{/M_1} = \tilde{\xi}$.

6.2. All the primary constraints are first class

As in Section 4.2, we take the almost product structure $(\mathcal{A}_1, \mathcal{B}_1)$ on M_1 adapted to the presymplectic form ω_1. Now, we fix a solution of the constrained dynamics by taking

$$\left(\bar{\mathcal{P}}(\mathcal{A}_1\left(X_{H/M_1}\right))\right)_{/\bar{M}_1} .$$

We have $\tilde{\xi} = \mathcal{A}_1\left(X_{H/M_1}\right)$.

6.3. There exist first class and second class constraints

In this case, we have three different almost product structures $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ on T^*Q and $(\mathcal{A}_1, \mathcal{B}_1)$ and $(\bar{\mathcal{P}}, \bar{\mathcal{Q}})$ on M_1. A solution of the dynamics is just

$$\left(\bar{\mathcal{P}}(\mathcal{A}_1(\widetilde{\mathcal{A}}(X_H)_{/M_1}))\right)_{/\bar{M}_1} .$$

We have $\tilde{\xi} = \mathcal{A}_1(\widetilde{\mathcal{A}}(X_H)_{/M_1})$.

Remark 6.1. If the vector field $\bar{\mathcal{P}}_1(\xi)_{/\bar{P}_1}$ is not a SODE then we can construct a submanifold $\bar{S}$ of $\bar{P}_1$ and a diffeomorphism $\alpha : \bar{M}_1 \longrightarrow \bar{S}$ such that $\xi_{\bar{S}} = T\alpha(\bar{\mathcal{P}}(\tilde{\xi})_{/\bar{M}_1})$ is a solution of the constrained dynamics which, moreover, verifies the SODE condition.

Remark 6.2. For an arbitrary degenerate Lagrangian system subjected to constraints it is possible to construct an algorithm similar to that constructed by Gotay and Nester (see [10] for details).

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