DIFFERENTIAL GEOMETRY AND APPLICATIONS
Proc. Conf., Aug. 28 – Sept. 1, 1995, Brno, Czech Republic
Masaryk University, Brno 1996, 611–624

VARIATIONAL SEQUENCE IN HIGHER ORDER MECHANICS

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ABSTRACT. The variational sequence in higher order mechanics is considered. Explicit coordinate formulas for factorization mappings and quotient mappings for the first three columns are determined which includes the total derivative, Euler – Lagrange and Helmholtz – Sonin mappings.

1. INTRODUCTION

One of the key problems of the calculus of variations is characterization of the Euler – Lagrange mapping, particularly its kernel and image. These problems can be solved very effectively by means of the so called variational sequence constructed as the quotient sequence obtained by factorization of well–known De Rham sequence of modules of differential forms defined on a finite jet prolongation of an underlying fibered manifold with respect to appropriately chosen subsequence.

The general concept of the variational sequence was presented by Krupka in his earlier works (see e.g. [1]). The problem was then completely solved by the same author for the first order mechanics [2]. The suitable subsequence of De Rham sequence was found and the corresponding quotient sequence was completely constructed. It was shown that first three quotient mappings are very closely related to the problem of triviality of a lagrangian, Euler – Lagrange mapping and variationality conditions, respectively.

In our paper the more general problem is treated: The properties of the variational sequence in higher order mechanics are studied. The calculation technique for obtaining the chart expressions representing the classes of differential forms in the factorization procedure is developed and the explicit coordinate formulas for factorization as well as for quotient mappings are obtained for first three columns of the corresponding sequence diagram. Finally, it is shown that in the general (i.e. higher order) case the first three quotient mappings can be interpreted in the same way as in the first–order case: For the first of them, E_0, the image of

1991 *Mathematics Subject Classification.* 49Fxx, 58Axx, 58Exx, 70Gxx.
Key words and phrases. Fibered manifold, r–jet prolongation, contact form, variational sequence, Lagrangian, Euler – Lagrange mapping, Helmholtz – Sonin conditions of variationality.
This paper is in final form and no version of it will be submitted for publication elsewhere.
The author was supported by grant 201/93/2245 of the Czech Grant Agency.

a function is the corresponding trivial lagrangian, E_1 is the well–known Euler – Lagrange mapping, and E_2 is very closely connected with the variationality conditions. The E_2–image of the class of variational 2–forms is represented by the zero form and the coefficients of the chart expression of the representative of each class belonging to Im E_2 are directly the left–hand sides of well–known Helmholtz – Sonin conditions.

Let us note that the same problem was studied by Štefánek [3] whose results on variationality are equivalent to ours.

2. Underlying structures and basic notations

We use the following standard notation through the paper. Y is a *fibered manifold* with the *one–dimensional base* X and projection π. Let us denote $m = \dim Y - 1$. For an arbitrary integer $s \geq 0$, j^sY is the *s – jet prolongation* of Y, π_s and $\pi_{s,r}$ for $s \geq r \geq 0$ are the canonical projections of j^sY on X and j^rY, respectively; $\dim j^sY = (s+1)m + 1$. Let $1 \leq \sigma \leq m$ and (V, ψ), $\psi = (t, q_0^\sigma)$, be a *fibered chart* on Y. Then we denote by (U, φ) and (V_s, ψ_s) the *associated chart* on X and *associated fibered chart* on j^sY, respectively, where $U = \pi(V)$, $\varphi = (t)$, $V_s = \pi_{s,0}^{-1}(V)$, $\psi_s = (t, q_0^\sigma, q_1^\sigma, \ldots, q_s^\sigma)$.

Let $\Omega_s^0 V$ be the ring of smooth functions on V_s. We denote by $\Omega_s^k V$ the $\Omega_s^0 V$*–module of smooth differential k–forms* on V_s. The following definitions are adapted for the practical purpose.

Let (V, ψ) be a fibered chart on Y and $\varrho \in \Omega_s^k V$ an arbitrary k–form with the following chart expression:

$$\varrho = \frac{1}{(k-1)!} \sum_{0 \leq i_1, \ldots, i_{k-1} \leq s} A\,{}^{i_1 \ldots i_{k-1}}_{\sigma_1 \ldots \sigma_{k-1}} dq_{i_1}^{\sigma_1} \wedge \cdots \wedge dq_{i_{k-1}}^{\sigma_{k-1}} \wedge dt +$$

$$+ \frac{1}{k!} \sum_{0 \leq i_1, \ldots, i_k \leq s} B\,{}^{i_1 \ldots i_k}_{\sigma_1 \ldots \sigma_k} dq_{i_1}^{\sigma_1} \wedge \cdots \wedge dq_{i_k}^{\sigma_k}. \tag{1}$$

Coefficients $A\,{}^{i_1 \ldots i_{k-1}}_{\sigma_1 \ldots \sigma_{k-1}}$, $B\,{}^{i_1 \ldots i_k}_{\sigma_1 \ldots \sigma_k}$ are completely antisymmetric in multiindices, $0 \leq i_1, \ldots, i_k \leq s$, $1 \leq \sigma_1, \ldots, \sigma_k \leq m$.

It can be easily shown (see e.g. [2]) that $\pi_{s+1,\,s}^* \varrho$ can be written in the form

$$\pi_{s+1,\,s}^* \varrho = p_{k-1} \varrho + p_k \varrho,$$

where

$$p_{k-1} \varrho = \frac{1}{(k-1)!} \sum_{0 \leq i_1, \ldots, i_{k-1} \leq s} \overline{A\,{}^{i_1 \ldots i_{k-1}}_{\sigma_1 \ldots \sigma_{k-1}}}\, \omega_{i_1}^{\sigma_1} \wedge \cdots \wedge \omega_{i_{k-1}}^{\sigma_{k-1}} \wedge dt\,, \tag{2}$$

$$\overline{A\,{}^{i_1 \ldots i_{k-1}}_{\sigma_1 \ldots \sigma_{k-1}}} = A\,{}^{i_1 \ldots i_{k-1}}_{\sigma_1 \ldots \sigma_{k-1}} + \sum_{j=0}^{s} B\,{}^{i_1 \ldots i_{k-1} j}_{\sigma_1 \ldots \sigma_{k-1} \nu}\, q_{j+1}^\nu\,,$$

$$p_k \varrho = \frac{1}{k!} \sum_{0 \le i_1, \ldots, i_k \le s} B\,{}^{i_1 \ldots i_k}_{\sigma_1 \ldots \sigma_k}\, \omega^{\sigma_1}_{i_1} \wedge \cdots \wedge \omega^{\sigma_k}_{i_k} \,,$$
$$\omega^\sigma_j = dq^\sigma_j - q^\sigma_{j+1}\, dt \; for \; 0 \le j \le s.$$

Coefficients $\overline{A\,{}^{i_1 \ldots i_{k-1}}_{\sigma_1 \ldots \sigma_{k-1}}}$ are again completely antisymmetric in multiindices. $p_{k-1}\varrho$ and $p_k\varrho$ represent the $(k-1)$–*contact* and k–*contact part* of the form ϱ, respectively. The 0–contact part of a form is usually called the *horizontal* one and the mapping $h : \Omega^k_s \ni \varrho \longrightarrow p_0\varrho = h\varrho \in \Omega^k_{s+1}$ is called *the horizontalization mapping.* The (nonzero) k–form ϱ is called k–*contact* if $p_{k-1}\varrho = 0$, or $(k-1)$–*contact* if $p_k\varrho = 0$. The 0–contact form is called *horizontal*.

3. Variational sequence

In the following we denote $\Omega^k_{s,c} \subset \Omega^k_s$ the subspace of k–contact k–forms and $\Theta^k_s = \Omega^k_{s,c} + d\,\Omega^{k-1}_{s,c}$ for $k \ge 1$. $d\,\Omega^{k-1}_{s,c}$ is the image of $\Omega^{k-1}_{s,c}$ by the exterior derivative operator d. For any open set $V \subset Y$ there is a unique decomposition of any form $\varrho \in \Theta^k_s\, V$ such that $\varrho = \varrho_0 + d\varrho'$, $\varrho_0 \in \Omega^k_{s,c}\, V$, $\varrho' \in \Omega^{k-1}_{s,c}\, V$ (see e.g. [2]).

Now let us construct the variational sequence. The diagram

$$0 \longrightarrow R \longrightarrow \Omega^0_s \longrightarrow \Omega^1_s \longrightarrow \Omega^2_s \longrightarrow \cdots \longrightarrow \Omega^{(s+1)m+1} \longrightarrow 0\,, \tag{3}$$

in which the first two arrows denote the canonical injections and the remaining ones mean the exterior derivative d, represents the well–known (exact) De Rham sequence. Consider its subsequence

$$0 \longrightarrow \Theta^1_s \longrightarrow \Theta^2_s \longrightarrow \cdots \longrightarrow \Theta^{sm+1}_s \longrightarrow 0\,. \tag{4}$$

It can be immediately seen that it is exact too. The corresponding quotient sequence

$$\begin{gathered} 0 \longrightarrow R \longrightarrow \Omega^0_s \xrightarrow{E_0} \Omega^1_s/\Theta^1_s \xrightarrow{E_1} \Omega^2_s/\Theta^2_s \xrightarrow{E_2} \cdots \\ \cdots \longrightarrow \Omega^{sm+1}_s/\Theta^{sm+1}_s \longrightarrow \Omega^{sm+2}_s \longrightarrow \cdots \longrightarrow \Omega^{(s+1)m+1}_s \longrightarrow 0 \end{gathered} \tag{5}$$

is exact as well and is called the *variational sequence of the s–th order*. The quotient mappings $E_k : \Omega^k_s/\Theta^k_s \longrightarrow \Omega^{k+1}_s/\Theta^{k+1}_s$ are defined by

$$E_k\,(\,[\varrho]\,) = [d\varrho]\,, \tag{6}$$

where $[\varrho]$ denotes the class of the form ϱ. The general properties of these mappings were studied elsewhere [1], [2]. The main purpose of our calculations is to obtain the explicit coordinate formulas for $[\varrho]$ and $E_k\,(\,[\varrho]\,)$. From the point of view of the variational theory the meaning of first three quotient mappings E_0, E_1, E_2 is evident. On the other hand, it appears that solving the problem of finding the coordinate expressions for factorizations and quotient mappings in the r–th order variational sequence, we must work on $s = (2r+1)$–jet prolongation of the underlying fibered manifold Y. Thus we examine the factorization mappings F_k and quotient mappings E_k generally in the following diagram (see the final note of Chapter 4) and we obtain the explicit coordinate formulas for the appropriately chosen representatives of classes $[\varrho] = F_k(\varrho)$ and $E_k\left([\varrho]\right) = [d\varrho]$ especially for a k–form ϱ defined primarily on $j^r Y$, i.e. $\left[\pi^*_{2r+1,\,r}\,\varrho\right]$ and $\left[\pi^*_{2r+1,\,r}\,d\varrho\right]$. In the following calculations we write $\varrho \sim \omega$ for $[\varrho] = [\omega]$. The fact that a form η is a representative of the class $[\varrho]$ we express by $[\varrho] \cong \eta$ (the class $[\varrho]$ is represented by η).

(7)

$$
\begin{array}{ccccccccc}
 & & & 0 & & 0 & & 0 & \\
 & & & \downarrow i_{0,1} & & \downarrow i_{0,2} & & \downarrow i_{0,3} & \\
 & 0 & \overset{i_{0,1}}{\rightarrow} & \Theta^1_{2r+1} & \overset{d_{1,c}}{\rightarrow} & \Theta^2_{2r+1} & \overset{d_{2,c}}{\rightarrow} & \Theta^3_{2r+1} & \rightarrow \cdots 0 \\
 & \downarrow i_0 & & \downarrow i_1 & & \downarrow i_2 & & \downarrow i_3 & \\
0 \overset{i_0}{\rightarrow} R \overset{i_1}{\rightarrow} & \Omega^0_{2r+1} & \overset{d_0}{\rightarrow} & \Omega^1_{2r+1} & \overset{d_1}{\rightarrow} & \Omega^2_{2r+1} & \overset{d_2}{\rightarrow} & \Omega^3_{2r+1} & \rightarrow \cdots 0 \\
 & & E_0 \searrow & \downarrow F_1 & & \downarrow F_2 & & \downarrow F_3 & \\
 & & & \Omega^1_{2r+1}/\Theta^1_{2r+1} & \overset{E_1}{\rightarrow} & \Omega^2_{2r+1}/\Theta^2_{2r+1} & \overset{E_2}{\rightarrow} & \Omega^3_{2r+1}/\Theta^3_{2r+1} & \rightarrow \cdots 0 \\
 & & & \downarrow o_1 & & \downarrow o_2 & & \downarrow o_3 & \\
 & & & 0 & & 0 & & 0 &
\end{array}
$$

d ... exterior derivative	i ... canonical injections
o ... zero mappings	F ... factorization mappings
E ... quotient mappings	$\Theta^k_{2r+1} = Ker\ F_k$

4. Factorization mappings

The idea of the calculation procedure which leads to the chart expresions of the representative of the class $[\varrho]$ of a k–form $\varrho \in \Omega^k_r V$ lies in finding the most suitable decomposition of the form $\pi^*_{s,\,r}\,\varrho$ with appropriately chosen $s \geq r$ as follows: $\pi^*_{s,\,r}\,\varrho = \varrho_0 + \varrho_\Theta$ where $\varrho_\Theta \in \Theta^k_s$ and ϱ_0 is the representative of the class of ϱ, i.e. $[\varrho] \cong \varrho_0$. The mentioned procedure is used in the proofs of the following theorem.

Theorem 1. *Let* $(V,\,\psi)$, $\psi = (t,\,q^\sigma_0)$, $1 \leq \sigma \leq m$, *be a fibered chart on* Y.

(a) *Let $\varrho \in \Omega^1_r V$ have the chart expression*

$$\varrho = A\, dt + \sum_{j=0}^{r} B^j_\sigma\, dq^\sigma_j\ .$$

Then the class of the form ϱ, $[\varrho] = F_1(\varrho)$, is defined on the $s = (r+1)$–jet prolongation of Y, i.e. $[\varrho] \in \Omega^1_{r+1}V/\Theta^1_{r+1}V$, by the coordinate formula for its representative:

$$[\varrho] \cong \overline{A}\, dt = (A + \sum_{j=0}^{r} B^j_\sigma\, q^\sigma_{j+1})\, dt\ .$$

(b) *Let $\varrho \in \Omega^2_r V$ have the chart expression*

$$\varrho = \sum_{j=0}^{r} A^j_\sigma\, dq^\sigma_j \wedge dt + \frac{1}{2} \sum_{i,j=0}^{r} B^{ji}_{\sigma\nu}\, dq^\sigma_j \wedge dq^\nu_i\ , \quad B^{ji}_{\sigma\nu} + B^{ij}_{\nu\sigma} = 0\ .$$

Then the class $[\varrho] = F_2(\varrho)$ is defined on $s = (2r+1)$–jet prolongation of Y, i.e. $[\varrho] \in \Omega^2_{2r+1}V/\Theta^2_{2r+1}V$, and its representative is given by the following coordinate formula:

$$[\varrho] \cong \sum_{j=0}^{r} (-1)^j\, \frac{d^j \overline{A^j_\sigma}}{dt^j}\, \omega^\sigma_0 \wedge dt\ , \quad \overline{A^j_\sigma} = A^j_\sigma + \sum_{i=0}^{r} B^{ji}_{\sigma\nu}\, q^\nu_{i+1}\ ,$$

in which d/dt denotes the total (formal) derivative.

(c) *Let $\varrho \in \Omega^3_r V$ have the chart expression*

$$\varrho = \frac{1}{2} \sum_{i,j=0}^{r} A^{ji}_{\sigma\nu}\, dq^\sigma_j \wedge dq^\nu_i \wedge dt + \frac{1}{3!} \sum_{i,j,k=0}^{r} B^{jik}_{\sigma\nu\lambda}\, dq^\sigma_j \wedge dq^\nu_i \wedge dq^\lambda_k\ ,$$

$A^{ji}_{\sigma\nu} + A^{ij}_{\nu\sigma} = 0$ and $B^{jik}_{\sigma\nu\lambda}$ are completely antisymmetric in multiindices. Then the class $[\varrho] = F_3(\varrho)$ is defined on $s = (2r+1)$–jet prolongation of Y, i.e. $[\varrho] \in \Omega^3_{2r+1}V/\Theta^3_{2r+1}V$, and its representative is given by the coordinate formula

$$[\varrho] \cong \sum_{k=0}^{2r} \mathcal{B}^k_{\sigma\nu} \omega^\sigma_k \wedge \omega^\nu_0 \wedge dt\ ,$$

where

$$\mathcal{B}^k_{\sigma\nu} = \frac{1}{2} \sum_{m=p}^{q} \sum_{i=m}^{r} (-1)^i \binom{i}{m} \frac{d^{i-m} \overline{A^{k-m,\,i}_{\sigma\nu}}}{dt^{i-m}}\ , \qquad \overline{A^{ji}_{\sigma\nu}} = A^{ji}_{\sigma\nu} + \sum_{l=0}^{r} B^{jil}_{\sigma\nu\lambda}\, q^\lambda_{l+1}$$

and $p = 0$, $q = k$ for $0 \le k \le r$, $p = k - r$, $q = r$ for $r < k \le 2r$.

Proof. Let $\varrho \in \Omega^k_r V$. We obtain the relation $[\varrho] = [p_{k-1}\varrho]$ as the immediate consequence of the definition of the class $[\varrho]$.

(a) For $k=1$ we have $[\varrho]=[p_0\varrho]=[h\varrho]$. For $\varrho = A\,dt + \sum_{j=0}^{r} B^j_\sigma\, dq^\sigma_j$ the relations (2) give the coordinate expression for $h\varrho \in \Omega^k_{r+1}V$ which is simultaneously the representative of the class $[\varrho]$:

$$[\varrho]=[h\varrho]\cong \overline{A}\,dt = (A+\sum_{j=0}^{r} B^j_\sigma\, q^\sigma_{j+1})\,dt\,. \tag{8}$$

We can see that $[\varrho]\in \Omega^1_{r+1}V/\Theta^1_{r+1}V$, because $\overline{A}\in\Omega^0_{r+1}V$.

(b) For $k=2$ and $\varrho = A^j_\sigma\, dq^\sigma_j \wedge dt \;+\; \sum_{i=0}^{r} B^{ji}_{\sigma\nu}\, dq^\sigma_j \wedge dq^\nu_i \;\in\; \Omega^2_r V$ we obtain by using the formulas (2):

$$p_1\varrho = \sum_{j=0}^{r} \overline{A^j_\sigma}\,\omega^\sigma_j \wedge dt\,, \qquad \overline{A^j_\sigma}\;+\;\sum_{i=0}^{r} B^{ji}_{\sigma\nu}\, q^\nu_{i+1}\in\Omega^0_{r+1}V\,.$$

For obtaining the representative of the class $[\varrho]=[p_1\varrho]$ we look for the appropriate decomposition of $\pi^*_{s,\,r+1}p_1\varrho$:

$$p_1\varrho = \overline{A^0_\sigma}\,\omega^\sigma_0\wedge dt \;+\;\sum_{j=1}^{r}\overline{A^j_\sigma}\,\omega^\sigma_j\wedge dt = \overline{A^0_\sigma}\,\omega^\sigma_0\wedge dt\;+\;\sum_{j=1}^{r}(-\overline{A^j_\sigma})\,d\omega^\sigma_{j-1}\,.$$

The last relation follows from the fact that $d\omega^\sigma_{j-1} \;=\; \pi^*_{r+1,\,r}d(dq^\sigma_{j-1}\;-\;-q^\sigma_j\,dt) \;=\; -\pi^*_{r+1,\,r}(dq^\sigma_j\wedge dt)\;=\;-\omega^\sigma_j\wedge dt$. The next calculation procedure is based on repeating the appropriate number of steps of the following type:

$$\pi^*_{r+2,\,r+1}(-\overline{A^j_\sigma}\,d\omega^\sigma_{j-1}) = \pi^*_{r+2,\,r+1}\left[\left(d\overline{A^j_\sigma}\wedge\omega^\sigma_{j-1}\right)\;-\;d(\overline{A^j_\sigma}\,\omega^\sigma_{j-1})\right]=$$

$$=-\frac{d\overline{A^j_\sigma}}{dt}\,\omega^\sigma_{j-1}\wedge dt\;+\;p_1(d\overline{A^j_\sigma})\wedge\omega^\sigma_{j-1}\;-\pi^*_{r+2,\,r+1}\,d(\overline{A^j_\sigma}\omega^\sigma_{j-1})\,.$$

The form $\pi^*_{r+2,\,r+1}(-\overline{A^j_\sigma}\,d\omega^\sigma_{j-1})$ is expressed as the sum of forms $-\frac{d\overline{A^j_\sigma}}{dt}\,\omega^\sigma_{j-1}\wedge dt\in \Omega^2_{r+2}V$, $p_1(d\overline{A^j_\sigma})\wedge\omega^\sigma_{j-1}\in\Omega^2_{r+2,\,c}V$ and $-\pi^*_{r+2,\,r+1}\,d(\overline{A^j_\sigma}\omega^\sigma_{j-1})\in\pi^*_{r+2,\,r+1}d\Omega^1_{r+1,\,c}V$ $\subset d\Omega^1_{r+2,\,c}V$ and thus

$$p_1\varrho\sim\overline{A^0_\sigma}\,\omega^\sigma_0\wedge dt\;+\;(-1)^1\sum_{j=1}^{r}\frac{d\overline{A^j_\sigma}}{dt}\,\omega^\sigma_{j-1}\wedge dt=$$

$$=\left(\overline{A^0_\sigma}\;+(-1)^1\frac{d\overline{A^1_\sigma}}{dt}\right)\omega^\sigma_0\wedge dt\;+\;(-1)^1\sum_{j=2}^{r}\frac{d\overline{A^j_\sigma}}{dt}\,\omega^\sigma_{j-1}\wedge dt=$$

$$=\left(\overline{A^0_\sigma}\;+(-1)^1\frac{d\overline{A^1_\sigma}}{dt}\right)\omega^\sigma_0\wedge dt\;+\;(-1)^2\sum_{j=2}^{r}\frac{d\overline{A^j_\sigma}}{dt}d\omega^\sigma_{j-2}\,.$$

Repeating the preceding procedure without writing the necessary pullbacks we finally obtain

(9)
$$[\varrho] \cong \left(\overline{A^0_\sigma} - \frac{d\overline{A^1_\sigma}}{dt} + \frac{d^2\overline{A^2_\sigma}}{dt^2} + \cdots + (-1)^r \frac{d^r\overline{A^r_\sigma}}{dt^r} \right) \omega^\sigma_0 \wedge dt = \sum_{j=0}^{r} (-1)^j \frac{d^j\overline{A^j_\sigma}}{dt^j} \omega^\sigma_0 \wedge dt \,.$$

It is evident that because of the expression $d^r(\overline{A^r_\sigma})/dt^r \in \Omega^0_{2r+1}V$ we get the resulting chart expression defined on the $s = (2r+1)$–jet prolongation of Y, i.e. $[\varrho] \in \Omega^2_{2r+1}V/\Theta^2_{2r+1}V$.

(c) Let $k = 3$ and $\varrho = \frac{1}{2}\sum_{i,\,j=0}^{r} A^{ji}_{\sigma\nu}\, dq^\sigma_j \wedge dq^\nu_i \wedge dt + \frac{1}{3!}\sum_{i,\,j,\,k=0}^{r} B^{jik}_{\sigma\nu\lambda}\, dq^\sigma_j \wedge \wedge dq^\nu_i \wedge dq^\lambda_k$. Then the relations (2) give $p_2\varrho = \frac{1}{2}\sum_{i,\,j=0}^{r} \overline{A^{ji}_{\sigma\nu}}\, \omega^\sigma_j \wedge \omega^\nu_i \wedge dt$, where $\overline{A^{ji}_{\sigma\nu}} = A^{ji}_{\sigma\nu} + \sum_{k=0}^{r} B^{jik}_{\sigma\nu\lambda} q^\lambda_{k+1}$ and $\overline{A^{ji}_{\sigma\nu}} + \overline{A^{ij}_{\nu\sigma}} = 0$. We try to write the representative of $[\varrho]$ in the form

$$[\varrho] \cong \sum_{k=0}^{s} \mathcal{B}^k_{\sigma\nu} \omega^\sigma_k \wedge \omega^\nu_0 \wedge dt \,, \tag{10}$$

in which s is the appropriate order of the jet prolongation of Y and $\mathcal{B}^k_{\sigma\nu}$, $1 \le \sigma,\, \nu \le m$, $0 \le k \le s$, are unknown coefficients.

Let F be a function and ω and η contact 1–forms on j^sY. Then we have, again without writing the necessary pullbacks,

$$d(F\omega \wedge \eta) = dF \wedge \omega \wedge \eta + F\, d\omega \wedge \eta - F\omega \wedge d\eta =$$
$$= \frac{dF}{dt}\omega \wedge \eta \wedge dt + p_1(dF) \wedge \omega \wedge \eta + F\, d\omega \wedge \eta - F\omega \wedge d\eta$$

and thus

$$F\omega\wedge d\eta = \frac{dF}{dt}\omega\wedge\eta\wedge dt + F\, d\omega\wedge\eta - d(F\omega\wedge\eta) + p_1(dF)\wedge\omega\wedge\eta \sim \frac{dF}{dt}\omega\wedge\eta\wedge dt + F\, d\omega\wedge\eta \,,$$

because $-d(F\omega \wedge \eta) \in d\Omega^2_{s,\,c}$ and $p_1(dF) \wedge \omega \wedge \eta \in \Omega^3_{s+1,\,c}$.

By using this relation for the expression $\overline{A^{ji}_{\sigma\nu}}\omega^\sigma_j \wedge \omega^\nu_i \wedge dt = -\overline{A^{ji}_{\sigma\nu}}\omega^\sigma_j \wedge d\omega^\nu_{i-1}$ in which we denote $\overline{A^{ji}_{\sigma\nu}} = F$, $\omega^\sigma_j = \omega$ and $\omega^\nu_{i-1} = \eta$ for $i \ge 1$, we obtain

$$\overline{A^{ji}_{\sigma\nu}}\omega^\sigma_j \wedge \omega^\nu_i \wedge dt = -\overline{A^{ji}_{\sigma\nu}}\omega^\sigma_j \wedge d\omega^\nu_{i-1} \sim -\frac{d\overline{A^{ji}_{\sigma\nu}}}{dt}\omega^\sigma_j \wedge \omega^\nu_{i-1} \wedge dt - \overline{A^{ji}_{\sigma\nu}} d\omega^\sigma_j \wedge \omega^\nu_{i-1} =$$

$$= -\frac{d\overline{A^{ji}_{\sigma\nu}}}{dt}\omega^\sigma_j \wedge \omega^\nu_{i-1} \wedge dt - \overline{A^{ji}_{\sigma\nu}}\omega^\sigma_{j+1} \wedge \omega^\nu_{i-1} \wedge dt = -\left(\frac{d\overline{A^{ji}_{\sigma\nu}}}{dt}\omega^\sigma_j + \overline{A^{ji}_{\sigma\nu}}\omega^\sigma_{j+1}\right) \wedge \omega^\nu_{i-1} \wedge dt =$$

$$= \begin{cases} -\left(\frac{d\overline{A^{j1}_{\sigma\nu}}}{dt}\omega^\sigma_j + \overline{A^{j1}_{\sigma\nu}}\omega^\sigma_{j+1}\right) \wedge \omega^\nu_0 \wedge dt & for\ \ i = 1 \\ \frac{d\overline{A^{ji}_{\sigma\nu}}}{dt}\omega^\sigma_j \wedge d\omega^\nu_{i-2} + \overline{A^{ji}_{\sigma\nu}}\omega^\sigma_{j+1} \wedge d\omega^\nu_{i-2} & for\ \ i \geq 2 \end{cases}$$

By repeating this procedure step by step we get

$$\overline{A^{ji}_{\sigma\nu}}\omega^\sigma_j \wedge \omega^\nu_i \wedge dt \sim \sum_{l=0}^{i}(-1)^i \binom{i}{l} \frac{d^{i-l}\,\overline{A^{ji}_{\sigma\nu}}}{dt^{i-l}}\,\omega^\sigma_{j+l} \wedge \omega^\sigma_0 \wedge dt$$

and thus

$$[\varrho] = [p_2\varrho] \cong \sum_{i,\,j=0}^{r} \frac{1}{2}(-1)^i \sum_{l=0}^{i} \binom{i}{l} \frac{d^{i-l}\overline{A^{ji}_{\sigma\nu}}}{dt^{i-l}}\,\omega^\sigma_{j+l} \wedge \omega^\nu_0 \wedge dt \tag{11}$$

for $j + l \in \{0, 1, \ldots, 2r\}$.

For $j + l = k \leq r$ the following possibilities realize: $j = 0, l = k$ for $i \in$ $\in \{k, \ldots, r\}$; $j = 1, l = k-1$ for $i \in \{k-1, \ldots, r\}$; $\ldots$; $j = m, l = k-m$ for $i \in \{k-m, \ldots, r\}$; $\ldots$; $j = k, l = 0$ for $i \in \{0, \ldots, r\}$. Then

$$\mathcal{B}^k_{\sigma\nu} = \frac{1}{2}\sum_{m=0}^{k}\sum_{i=m}^{r}(-1)^i\binom{i}{m}\frac{d^{i-m}\overline{A^{k-m,\,i}_{\sigma\nu}}}{dt^{i-m}} \quad for\ \ 0 \leq k \leq r\ , \tag{12}$$

and, by the completely analogous way,

$$\mathcal{B}^k_{\sigma\nu} = \frac{1}{2}\sum_{m=k-r}^{r}\sum_{i=m}^{r}(-1)^i\binom{i}{m}\frac{d^{i-m}\overline{A^{k-m,\,i}_{\sigma\nu}}}{dt^{i-m}} \quad for\ \ r+1 \leq k \leq 2r\ . \tag{13}$$

It can be again seen that $\mathcal{B}^k_{\sigma\nu}$ are defined on $j^{2r+1}Y$. The resulting coordinate expression for the representative of $[\varrho] \in \Omega^3_{2r+1}V/\Theta^3_{2r+1}V$ is as follows:

$$\begin{aligned} [\varrho] \cong & \sum_{k=0}^{r}\left(\frac{1}{2}\sum_{m=0}^{k}\sum_{i=m}^{r}(-1)^i\binom{i}{m}\frac{d^{i-m}\overline{A^{k-m,\,i}_{\sigma\nu}}}{dt^{i-m}}\right)\omega^\sigma_k \wedge \omega^\nu_0 \wedge dt\ + \\ & + \sum_{k=r+1}^{2r}\left(\frac{1}{2}\sum_{m=k-r}^{r}\sum_{i=m}^{r}(-1)^i\binom{i}{m}\frac{d^{i-m}\overline{A^{k-m,\,i}_{\sigma\nu}}}{dt^{i-m}}\right)\omega^\sigma_k \wedge \omega^\nu_0 \wedge dt\ . \end{aligned} \tag{14}$$

It is evident that the above used calculation procedure ensures $[\varrho] \cong 0$ if and only if $\varrho \in \Theta^k_r$.

Note. Let in general $\eta \in \Omega^k_{2r+1} V$. Then there is a unique decomposition $\eta = \varrho + \eta'$ in which $\varrho \in \pi^*_{2r+1,\,r}\Omega^k_r$ and

$$\eta' = \sum_{0 \leq j_1, \dots, j_{k-1} \leq 2r+1} C^{\,j_1 \dots j_{k-1}}_{\,\sigma_1 \dots \sigma_{k-1}}\, dq^{\sigma_1}_{j_1} \wedge \dots \wedge dq^{\sigma_{k-1}}_{j_{k-1}} \wedge dt + \\ + \sum_{0 \leq j_1, \dots, j_k \leq 2r+1} D^{\,j_1 \dots j_k}_{\,\sigma_1 \dots \sigma_k}\, dq^{\sigma_1}_{j_1} \wedge \dots \wedge dq^{\sigma_k}_{j_k}\,,$$

where none of the forms $C^{\,j_1 \dots j_{k-1}}_{\,\sigma_1 \dots \sigma_{k-1}}\; dq^{\sigma_1}_{j_1} \wedge \dots \wedge dq^{\sigma_{k-1}}_{j_{k-1}} \wedge dt$, $D^{\,j_1 \dots j_k}_{\,\sigma_1 \dots \sigma_k}\; dq^{\sigma_1}_{j_1} \wedge \wedge \dots \wedge dq^{\sigma_k}_{j_k}$ is $\pi^*_{2r+1,\,r}$–projectable. Let η_0 be an arbitrarily choosen representative of the class $[\eta'] \in \Omega^k_{2r+1} V / \Theta^k_{2r+1} V$. We define $F_k(\eta) = [\eta] = [\varrho] + [\eta']$ or $[\eta] \cong \cong \varrho_0 + \eta_0$, where ϱ_0 is the representative of the class $[\varrho]$ given for $k = 1, 2, 3$ by Theorem 1. It can be cleary seen that $\Theta^k_{2r+1} V = Ker\ F_k$.

5. Quotient mappings

The explicit coordinate formulas for quotient mappings E_0, E_1, E_2 are obtained from the definition of E_k (6) by the direct application of relations (8), (9) and (10) on the form $d\varrho$ for $\varrho = f \in \Omega^0_r V$, $\varrho \in \Omega^1_r V$ and $\varrho \in \Omega^2_r V$, respectively. The results are summarized in the following theorem:

Theorem 2. *Let (V, ψ), $\psi = (t, q^\sigma_0)$, $1 \leq \sigma \leq m$, be a fibered chart on Y.*

(a) *Let $f \in \Omega^0_r V$. Then E_0 is defined on $j^{r+1} Y$, i.e. $E_0 : \Omega^0_r V \ni [f] \to \to E_0([f]) = [df] \in \Omega^1_{r+1} V / \Theta^1_{r+1} V$ and for the representative of $E_0([f])$ it holds*

$$E_0([f]) = [df] \cong \left(\frac{df}{dt}\right) dt\,.$$

(b) *Let $\varrho \in \Omega^1_r V$, $\varrho = P\, dt + \sum_{j=0}^r Q^j_\sigma dq^\sigma_j$. Then E_1 is defined on $j^{2r+1} Y$ i.e. $E_1 : \Omega^1_{2r+1} V / \Theta^1_{2r+1} V \ni [\varrho] \to E_1([\varrho]) = [d\varrho] \in \Omega^2_{2r+1} V / \Theta^2_{2r+1} V$, and for the representative of $E_1([\varrho])$ it holds*

$$E_1([\varrho]) \cong \varepsilon_\sigma \omega^\sigma_0 \wedge dt = \sum_{j=0}^r (-1)^i \frac{d^j}{dt^j}\left(\frac{\partial \overline{P}}{\partial q^\sigma_j} - \frac{dQ^j_\sigma}{dt} - Q^{j-1}_\sigma\right) =$$

$$= \left(\sum_{j=0}^r \left[(-1)^j \frac{d^j}{dt^j}\left(\frac{\partial \overline{P}}{\partial q^\sigma_j}\right)\right] + (-1)^{r+1} \frac{d^{r+1} Q^r_\sigma}{dt^{r+1}}\right) \omega^\sigma_0 \wedge dt\,,$$

where $\overline{P} = P + \sum_{j=0}^{r} Q_\sigma^j q_{j+1}^\sigma$ *and we put* $Q_\sigma^{j-1} = 0$ *for* $j = 0$.
(c) *Let* $\varrho \in \Omega_r^2 V$, $\varrho = \sum_{j=0}^{r} P_\sigma^j dq_j^\sigma \wedge dt + \frac{1}{2}\sum_{i,j=0}^{r} Q_{\sigma\nu}^{ji} dq_j^\sigma \wedge dq_i^\nu$ *. Then* E_2 *is defined on* $j^{2r+1}Y$*, i.e.* $E_2 : \Omega_{2r+1}^2 V/\Theta_{2r+1}^2 V \ni [\varrho] \to E_2([\varrho]) = [d\varrho] \in \Omega_{2r+1}^3 V/\Theta_{2r+1}^3 V$ *, and for the representative of* $E_2([\varrho])$ *it holds:*

$$E_2([\varrho]) \cong \sum_{k=0}^{2r} \mathcal{B}_{\sigma\nu}^k \omega_k^\sigma \wedge \omega_0^\nu \wedge dt \,,$$

where

$$\mathcal{B}_{\sigma\nu}^k = \frac{1}{2}\sum_{j=p}^{q}\sum_{i=k-j}^{r}(-1)^i \binom{i}{k-j}\frac{d^{i+j-k}}{dt^{i+j-k}}\left[\frac{\partial \overline{P_\nu^i}}{\partial q_j^\sigma} - \frac{\partial \overline{P_\sigma^j}}{\partial q_i^\nu} + \frac{dQ_{\sigma\nu}^{ji}}{dt} + Q_{\sigma\nu}^{j,\,i-1} - Q_{\nu\sigma}^{i,\,j-1}\right],$$

$p = 0$ *and* $q = k$ *for* $0 \le k \le r$, $p = k - r$ *and* $q = r$ *for* $r + 1 \le k \le 2r$, $\overline{P_\sigma^j} = P_\sigma^j + \sum_{i=0}^{r} Q_{\sigma\nu}^{ji} q_{i+1}^\nu$ *and we put* $Q_{\sigma\nu}^{u,\,v-1} = 0$ *for* $v = 0$.

Proof. (a) For $f \in \Omega_r^0 V$ it holds $E_0([f]) = [df] = [hdf]$ and thus, by using the formula (8) and taking into account that $[f] \cong f$ we obtain

$$E_0([f]) \cong \left(\frac{df}{dt}\right) dt \,. \tag{15}$$

It is evident that the representative of $E_0([f])$ is defined on $j^{r+1}Y$.

(b) For $\varrho \in \Omega_r^1 V$ and $\varrho = P\,dt + \sum_{j=0}^{r} Q_\sigma^j dq_j^\sigma$ the representative of $[\varrho]$ is given by $[\varrho] \cong \overline{P}dt = \left(P + \sum_{j=0}^{r} Q_\sigma^j q_{j+1}^\sigma\right) dt$. On the other hand, the exterior derivative of ϱ is as follows

$$d\varrho = \sum_{j=0}^{r}\left(\frac{\partial P}{\partial q_j^\sigma} - \frac{\partial Q}{\partial t}\right) dq_j^\sigma \wedge dt + \sum_{i,j=0}^{r}\frac{\partial Q_\nu^i}{\partial q_j^\sigma} dq_j^\sigma \wedge dq_i^\nu \,.$$

After antisymmetrization of the second sum we use the relations (2) with $k = 2$ (for obtaining $p_1 d\varrho$) and the formula (9). This gives the following result:

$$E_1([\varrho]) = [d\varrho] \cong \sum_{j=0}^{r}(-1)^j\frac{d^j}{dt^j}\left(\frac{\partial \overline{P}}{\partial q_j^\sigma} - \frac{dQ_\sigma^j}{dt} - Q_\sigma^{j-1}\right)\omega_0^\sigma \wedge dt =$$

$$= \left[\sum_{j=0}^{r}(-1)^j\frac{d^j}{dt^j}\left(\frac{\partial \overline{P}}{\partial q_j^\sigma}\right) + (-1)^{r+1}\frac{d^{r+1}Q_\sigma^r}{dt^{r+1}}\right]\omega_0^\sigma \wedge dt \,, \tag{16}$$

where we have put $Q_\sigma^{j-1} = 0$ for $j = 0$. The representative of $E_1([\varrho])$ is defined od $j^{2r+1}Y$.

(c) Let $\varrho \in \Omega^2_r V$ is expressed by $\varrho = \sum_{j=0}^{r} P^j_\sigma dq^\sigma_j \wedge dt + \frac{1}{2}\sum_{i,j=0}^{r} Q^{ji}_{\sigma\nu} dq^\sigma_j \wedge dq^\nu_i$. Then

$$[\varrho] \cong \left(\sum_{j=0}^{r}(-1)^j \frac{d^j \overline{P^j_\sigma}}{dt^j}\right) \omega^\sigma_0 \wedge dt \, , \; \overline{P^j_\sigma} = P^j_\sigma + \sum_{i=0}^{r} Q^{ji}_{\sigma\nu} q^\nu_{i+1} \, .$$

It holds, after antisymmetrization:

$$d\varrho = \frac{1}{2}\sum_{i,j=0}^{r}\left(\frac{\partial P^i_\nu}{\partial q^\sigma_j} - \frac{\partial P^j_\sigma}{\partial q^\nu_i} + \frac{1}{2}\frac{\partial Q^{ji}_{\sigma\nu}}{\partial t}\right) dq^\sigma_j \wedge dq^\nu_i \wedge dt +$$

$$+ \frac{1}{3!}\sum_{i,j,k=0}^{r}\left(\frac{\partial Q^{ik}_{\nu\lambda}}{\partial q^\sigma_j} + \frac{\partial Q^{ji}_{\sigma\nu}}{\partial q^\lambda_k} + \frac{\partial Q^{kj}_{\lambda\sigma}}{\partial q^\nu_i}\right) dq^\sigma_j \wedge dq^\nu_i \wedge dq^\lambda_k \, .$$

By denoting $A^{ji}_{\sigma\nu} = \left(\frac{\partial P^i_\nu}{\partial q^\sigma_j} - \frac{\partial P^j_\sigma}{\partial q^\nu_i} + \frac{1}{2}\frac{\partial Q^{ji}_{\sigma\nu}}{\partial t}\right)$, $B^{jik}_{\sigma\nu\lambda} = \left(\frac{\partial Q^{ik}_{\nu\lambda}}{\partial q^\sigma_j} + \frac{\partial Q^{ji}_{\sigma\nu}}{\partial q^\lambda_k} + \frac{\partial Q^{kj}_{\lambda\sigma}}{\partial q^\nu_i}\right)$ and using (2) for $k = 3$ and (11), we get

$$E_2([\varrho]) = [d\varrho] \cong \sum_{k=0}^{2r} \mathcal{B}^k_{\sigma\nu}\omega^\sigma_k \wedge \omega_0 \wedge dt \, ,$$

with

(17)

$$\mathcal{B}^k_{\sigma\nu} = \frac{1}{2}\sum_{j=p}^{q}\sum_{i=k-j}^{r}(-1)^i \binom{i}{k-j}\frac{d^{i+j-k}}{dt^{i+j-k}}\left[\frac{\partial \overline{P^i_\nu}}{\partial q^\sigma_j} - \frac{\partial \overline{P^j_\sigma}}{\partial q^\nu_i} + \frac{dQ^{ji}_{\sigma\nu}}{dt} + Q^{j,\,i-1}_{\sigma\nu} - Q^{i,\,j-1}_{\nu\sigma}\right] ,$$

where $p = 0$, $q = k$ for $0 \le k \le r$; $p = k - r$, $q = r$ for $r + 1 \le k \le 2r$ and $Q^{u,v-1}_{\sigma\nu} = 0$ for $v = 0$. We can see that the representative of $E_2([\varrho])$ is defined again on $j^{2r+1}Y$.

The relation of our results obtained for the quotient mappings to the commonly used objects of higher order mechanics is clear from the following theorem:

Theorem 3. *Let (V, ψ), $\psi = (t, q^\sigma_0)$, $1 \le \sigma \le m$, be a fibered chart on Y.*

(a) *Let $\varrho \in \Omega^0_{r-1}V$. Then the representative of $E_0([f])$ is the trivial lagrangian of the order r.*

(b) *Let $\varrho \in \Omega^1_r V$ and let the representative of $[\varrho]$ be a lagrangian of the order r. Then the representative $E_1([\varrho])$ is its Euler – Lagrange form.*

(c) Let $\varrho \in \Omega^2_r V$ and $[\varrho] \in \Omega^2_{2r} V / \Theta^2_{2r} V$, i.e. the representative of the class $[\varrho]$ is the $\pi_{2r,0}$–horizontal form defined on $j^{2r} Y$. Then ϱ is locally variational iff $E_2([\varrho]) = 0$.

Proof. (a) Let $f \in \Omega^0_{r-1} V$. Then, after (15), $E_0([f]) \cong \frac{df}{dt} dt$, where $\overline{P} = \frac{df}{dt} \in \Omega^0_r$. Thus $\lambda = \frac{df}{dt} dt$ is the trivial lagrangian because its Lagrange function has the form of the total derivative of a function defined on $j^{r-1} Y$.

(b) Let $\varrho \in \Omega^1_r V$ and $\varrho = P\, dt + \sum_{j=0}^r Q^j_\sigma dq^\sigma_j$. The representative of $[\varrho]$ is the lagrangian of the r–th order iff $Q^r_\sigma = 0$. Then $[\varrho] \cong \lambda = \overline{P}\, dt = \left(P + \sum_{j=0}^{r-1} Q^j_\sigma q^\sigma_{j+1} \right) dt$. The formula (16) gives

$$E_1([\varrho]) \cong \left[\sum_{j=0}^{r} (-1)^j \frac{d^j}{dt^j} \left(\frac{\partial \overline{P}}{\partial q^\sigma_j} \right) \right] \omega^\sigma_0 \wedge dt\,,$$

which is the usual expression for the Euler – Lagrange form of λ.

(c) Let $\varrho \in \Omega^2_r V$ and $\varrho = \sum_{j=0}^r P^j_\sigma dq^\sigma_j \wedge dt + \frac{1}{2} \sum_{i,j=0}^r Q^{ji}_{\sigma\nu} dq^\sigma_j \wedge dq^\nu_i$. The relation (9) leads to the following result:

$$[\varrho] \cong \sum_{j=0}^{r} (-1)^j \frac{d^j \overline{P^j_\sigma}}{dt^j} \omega^\sigma_j \wedge dt\,, \quad \textit{where} \quad \overline{P^j_\sigma} = P^j_\sigma + \sum_{i=0}^{r} Q^{ji}_{\sigma\nu} q^\nu_{i+1}\,.$$

It can be easily seen that $[\varrho] \in \Omega^2_{2r} V / \Theta^2_{2r} V$ iff $Q^{rr}_{\sigma\nu} = 0$ for $1 \le \sigma, \nu \le m$.

Let us now denote

$$\varepsilon_\sigma = \sum_{j=0}^{r} (-1)^j \frac{d^j \overline{P^j_\sigma}}{dt^j} \tag{18}$$

and

$$\mathcal{H}^k_{\sigma\nu} = \frac{\partial \varepsilon_\nu}{\partial q^\sigma_k} - (-1)^k \frac{\partial \varepsilon_\sigma}{\partial q^\nu_k} - \sum_{l=k+1}^{2r} (-1)^l \binom{l}{k} \frac{d^{l-k}}{dt^{l-k}} \left(\frac{\partial \varepsilon_\sigma}{\partial q^\nu_l} \right) \tag{19}$$

the Helmholtz – Sonin expressions commonly used in higher order mechanics. By using the expression (17) for cooeficients $\mathcal{B}^k_{\sigma\nu}$ and the relation

$$\frac{\partial}{\partial q^\nu_i} \left(\frac{d^k f}{dt^k} \right) = \sum_{l=0}^{k} \binom{k}{l} \frac{d^l}{dt^l} \left(\frac{\partial f}{\partial q^\nu_{i-k+l}} \right)\,,$$

in which f is a function and $\frac{\partial f}{\partial q^{\nu}_{i-k+l}} \equiv 0$ formally for $i-k+l<0$, we obtain, after some rather technical calculations, $\mathcal{B}^k_{\sigma\nu} = \frac{1}{2}\mathcal{H}^k_{\sigma\nu}$.

Example. For making our results more transparent we use them for the first order case. Let $r=1$ and (V,ψ), $\psi=(t,q^\sigma_0)$, $(\pi_0^{-1}(V),\psi_1)$, $\psi_1=(t,q^\sigma_0,q^\sigma_1)$ are the fibered chart on Y and the associated fibered chart on j^1Y, respectively.

(a) Let $f=f(t,q^\sigma_0,q^\sigma_1)\in\Omega^0_1 V$. Then

$$E_0([f]) \cong \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^\sigma_0} q^\sigma_1 + \frac{\partial f}{\partial q^\sigma_1} q^\sigma_2\right) dt$$

(b) Let $\varrho = P\,dt + \sum_{j=0}^{1} Q^j_\sigma dq^\sigma_j = P\,dt + Q^0_\sigma dq^\sigma_0 + Q^1_\sigma dq^\sigma_1$. $[\varrho] \cong \overline{P}\,dt = = \left(P + Q^0_\sigma q^\sigma_1 + Q^1_\sigma q^\sigma_2\right) dt$ has the form of the lagrangian of the order 2.

$$E_1([f]) \cong \left[\frac{\partial \overline{P}}{\partial q^\sigma_0} - \frac{d}{dt}\left(\frac{\partial \overline{P}}{\partial q^\sigma_1}\right) + \frac{d^2 Q^1_\sigma}{dt^2}\right] \omega^\sigma_0 \wedge dt =$$

$$= \left[\frac{\partial \overline{P}}{\partial q^\sigma_0} - \frac{d}{dt}\left(\frac{\partial \overline{P}}{\partial q^\sigma_1}\right) + \frac{d^2}{dt^2}\left(\frac{\partial \overline{P}}{\partial q^\sigma_2}\right)\right] \omega^\sigma_0 \wedge dt\ .$$

(c) Let $\varrho = P^0_\sigma dq^\sigma_0 \wedge dt + P^1_\sigma dq^\sigma_1 \wedge dt + \frac{1}{2}Q^{00}_{\sigma\nu} dq^\sigma_0 \wedge dq^\nu_0 + \frac{1}{2}Q^{10}_{\sigma\nu} dq^\sigma_1 \wedge dq^\nu_0 + +\frac{1}{2}Q^{01}_{\sigma\nu} dq^\sigma_0 \wedge dq^\nu_1 + \frac{1}{2}Q^{11}_{\sigma\nu} dq^\sigma_1 \wedge dq^\nu_1$.

Denote $\overline{P^0_\sigma} = P^0_\sigma + Q^{00}_{\sigma\nu} q^\nu_1 + Q^{01}_{\sigma\nu} q^\nu_2$, $\quad \overline{P^1_\sigma} = P^1_\sigma + Q^{10}_{\sigma\nu} q^\nu_1 + Q^{11}_{\sigma\nu} q^\nu_2$. Then

$$[\varrho] \cong \left(\overline{P^0_\sigma} - \frac{d}{dt}\overline{P^1_\sigma}\right) \omega^\sigma_0 \wedge dt = \Big[\left(P^0_\sigma + Q^{00}_{\sigma\nu} q^\nu_1 + Q^{01}_{\sigma\nu} q^\nu_2\right) -$$

$$- \left(\frac{dP^1_\sigma}{dt} + \frac{dQ^{10}_{\sigma\nu}}{dt} q^\nu_1 + \frac{dQ^{11}_{\sigma\nu}}{dt} q^\nu_2\right) - Q^{10}_{\sigma\nu} q^\nu_2 - Q^{11}_{\sigma\nu} q^\nu_3\Big] \omega^\sigma_0 \wedge dt$$

is defined on j^3Y, and

$$\varepsilon_\sigma = \left(P^0_\sigma - \frac{dP^1_\sigma}{dt}\right) + \left(Q^{00}_{\sigma\nu} - \frac{dQ^{10}_{\sigma\nu}}{dt}\right) q^\nu_1 + \left(Q^{01}_{\sigma\nu} - Q^{10}_{\sigma\nu} - \frac{dQ^{11}_{\sigma\nu}}{dt}\right) q^\nu_2 - Q^{11}_{\sigma\nu} q^\nu_3\ .$$

Then

$$E_2([\varrho]) = \mathcal{B}^0_{\sigma\nu} \omega^\sigma_0 \wedge \omega^\nu_0 \wedge dt + \mathcal{B}^1_{\sigma\nu} \omega^\sigma_1 \wedge \omega^\nu_0 \wedge dt + \mathcal{B}^2_{\sigma\nu} \omega^\sigma_2 \wedge \omega^\nu_0 \wedge dt\ ,$$

where (see (17))

$$\mathcal{B}^0_{\sigma\nu} = \frac{1}{2}\sum_{i=0}^{1}(-1)^i \frac{d^i}{dt^i}\left(\frac{\partial \overline{P^i_\nu}}{\partial q^\sigma_0} - \frac{\partial \overline{P^0_\sigma}}{\partial q^\nu_i}\right) - \frac{1}{2}\frac{d^2 Q^{01}_{\sigma\nu}}{dt^2}\ ,$$

$$\mathcal{B}^1_{\sigma\nu} = \frac{1}{2}\sum_{i=0}^{1}(-1)^i \frac{d^i}{dt^i}\left(\frac{\partial \overline{P^i_\nu}}{\partial q^\sigma_1} - \frac{\partial \overline{P^1_\sigma}}{\partial q^\nu_i}\right) - \frac{1}{2}\left(\frac{\partial \overline{P^1_\nu}}{\partial q^\sigma_0} - \frac{\partial \overline{P^0_\sigma}}{\partial q^\nu_1}\right) - \frac{dQ^{01}_{\sigma\nu}}{dt} - \frac{1}{2}\frac{d^2 Q^{11}_{\sigma\nu}}{dt^2}\,,$$

$$\mathcal{B}^2_{\sigma\nu} = -\frac{1}{2}\left[\left(\frac{\partial \overline{P^1_\nu}}{\partial q^\sigma_1} - \frac{\partial \overline{P^1_\sigma}}{\partial q^\nu_1}\right) + Q^{01}_{\sigma\nu} + Q^{10}_{\sigma\nu} + \frac{dQ^{11}_{\sigma\nu}}{dt}\right].$$

These results are the same as those obtained for $r = 1$ by Krupka [2]. For $Q^{11}_{\sigma\nu} = 0$, i.e. $\varepsilon_\sigma \in \Omega^0_2 V$, it can be seen after some calculations that $\mathcal{B}^k_{\sigma\nu} = \frac{1}{2}\mathcal{H}^k_{\sigma\nu}$ for $k = 0, 1, 2$.

6. Conclusions

From the Theorem 3 it follows that the variational sequence is in a very close relation to the commonly used objects of calculus of variations in higher order mechanics. Thus we can formulate the following generalization of usual definitions of calculus of variations:

representative of $[\varrho] \in \Omega^1_r$ ...	lagrangian associated with ϱ
representative of $[\varrho] \in \Omega^2_r$ ...	Euler – Langrange form of ϱ
representative of $[\varrho] \in \Omega^3_r$ ...	Helmholtz – Sonin form of ϱ
E_1	Euler – Lagrange mapping
E_2	Helmholtz – Sonin mapping

There are some open problems:

- the application of the presented calculation technique for obtaining the mappings F_k and E_{k-1} for $k \geq 4$,
- the invariance of the obtained coordinate expressions,
- the problem of the interpretation of mappings F_k and F_{k-1} for $k \geq 4$ from the point of view of calculus of variations.

I would like to thank Professor Demeter Krupka for his interest in my work and for many helpful discussions.

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