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A GENERALIZED APPROACH TO THE CONSERVATION LAWS PROBLEM IN GENERAL RELATIVITY

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Abstract. It is shown that Einstein pseudotensor and Møller tetrad complex of energy–momentum of gravitational filed represent special cases of more general construction.

1. Introduction

The conservation laws problem (i.e. the problem of definition of energy and momentum of gravitational filed) in general Relativity has an old history [1]. In this history, two solutions played an essential role; the original Einstein solution connected with his conception of general Relativity [2] and the Møller solution connected with introduction of new variables — tetrad fields — as an instrument of description of gravitation [3].

These solutions have some common properties. They are based on exclusion of second derivatives from the Lagrange density. Consequently, related quantities fulfilling differential conservation laws contain only derivatives of first degree . This guarantees a "reasonable" behavior of integral conserved quantities.

There are also differences. The Møller solution works with greater number of variables — 16 components of tetrad's instead of 10 components of metric. Besides the coordinate transformation, also gauge transformations of tetrads are essential in the Møller formulation. The Møller lagrangian and related conserved currents are invariant with respect to coordinate transformations, but not with respect to the gauge transformations.

In this work, we shall show that further enlargement of number of variables allows us to represent the Einstein and Møller solution as special cases of more general expressions of conserved quantities.

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2. Variables and lagrangians

The metric field g on the four dimensional space–time manifold can be expressed as

$$g = g_{(ab)} e^{(a)} e^{(b)} , \tag{1}$$

where $e^{(a)}$ represent four independent differential forms and $g_{(ab)}$ components of metric with respect to the basis $e^{(a)}$ (which is, generally speaking, anholonomic). In General Relativity $g_{(ab)}$ describes the gravitational field, whereas $e^{(a)}$ represents the system of reference in a most general sense. Consequently, we can take $g_{(ab)}$, $e_i^{(a)}$ as a collection of variables describing the gravitational field together with the system of reference. (Here, indices in bracket are related to the basis $e^{(i)}$; indices out of bracket are related to a coordinate system (which has, in general), no connection to the system of reference.) We shall also use the "dual" basis

$$\delta_i^j = e_i^{(a)} e_{(a)}^j , \quad e_{(b)\,i} = g_{(ab)} e_i^{(a)} \tag{2}$$

and the coordinate components of metric

$$g_{ik} = e_i^{(a)} e_k^{(b)} g_{(ab)} . \tag{3}$$

It is well–known that the Lagrange function in General Relativity is proportional to the scalar curvature R which can se expressed as a function of "reper" $e_k^{(a)}$ and its first and second derivatives, but the second derivatives can be separated in the "divergence" term. Hence

$$R = G + A^i_{;i} , \tag{4}$$

where

$$G = e^i_{(a);i} e^{(a)k}_{;k} - e^i_{(a);k} e^{(a)k}_{;i} \tag{5}$$

(semicolon means the covariant derivative).

In the coordinate formulation of General Relativity, the role of reper is played by the natural basis connected with coordinates. Therefore

$$e_i^{(a)} = \delta_i^a , \quad g_{ab} = g_{(ab)} . \tag{6}$$

Only the components of metric field are variables, whereas the components of basis are constant. In this case

$$-\frac{1}{2\kappa}\sqrt{-g}\, G\,(g_{ik}, g_{ik,l}) = \sqrt{-g}\, L \tag{7}$$

is the Lagrange density leading to the Einstein formulation of conservation laws.

On the other hand, in the Møller tetrad formulation we have

$$e_i^{(a)} = h_i^{(a)} , \quad g_{(ab)} = \eta_{ab} . \tag{8}$$

Here η_{ab} is Minkowski matrix and $h_i^{(a)}$ orthonormal reper — the tetrad. In this case, tetrad components are variables, whereas metric is constant. Corresponding Lagrange density is

$$-\frac{1}{2\kappa} \mid h \mid \; G\left(h_k^{(a)},\, h_{k\,,\,j}^{(a)}\right) = \mid h \mid \; L\,. \tag{9}$$

where h is determinant of tetrad $h_a^{(i)}$.

This situation suggests that it could be possible to comprehend Einstein and Møller formulation as some special examples of more general representation of Lagrange density, namely

$$-\frac{1}{2\kappa}\sqrt{-\,\tilde{g}}\;\mid e \mid\; G\left(e_k^{(a)}, e_{k,j}^{(a)}, g_{(ab)}, g_{(ab),k}\right) = \sqrt{-\,\tilde{g}}\;\mid e \mid\; L\,, \tag{10}$$

where e is determinant of $e_i^{(a)}$, $\tilde{g}$ is determinant of $g_{(ab)}$, $g = \tilde{g}\; e^2$ according to (3).

Let us study the consequences of this choice for the conservation laws formulation.

3. Review of general formulas

We shall use standard results connected with the first and second theorem of E. Noether. In the case of relativistic theories, it is possible to express all formulas in manifestly tensorial form [4]. Let us give a review of a procedure necessary for a calculation of conserved quantities.

It is clear that every vector field $X^i\,(x^j)$ on the space–time manifold generates a transformation of invariance of the Lagrange function $L\left(e_k^{(a)}, e_{k,j}^{(a)}, g_{(ab)}, g_{(ab),k}\right)$. Then X^i is related to the weakly (as a consequence of Lagrange functions) conserved vector field t^i and to the strongly (identically) conserved vector field S^i, where

$$t^i = -L\,X^i - \frac{\partial L}{\partial e_{k,i}^{(a)}}\,\overline{Y}_k^{(a)} - \frac{\partial L}{\partial g_{(pq),i}}\,\overline{Y}_{(pq)} \tag{11}$$

$$S^i = t^i - G^i\,;\;\; G^i = G^{ik}\,X_k \tag{12}$$

G^{ik} represents geometrical quantities equal to $-T^{ik}$ (energy–momentum tensor of matter) in the Einstein equations. $\overline{Y}_k^{(a)}$, $\overline{Y}_{(pq)}$ are Lie generators of components of reper $e_k^{(a)}$ and metric components $g_{(pq)}$. It holds for them

$$\overline{Y}_k^{(a)} = e_i^{(a)}\,X^i_{;k} - e_{k\,;i}^{(a)}X^i = \left[e^{(a)}, X\right]_k\,, \tag{13}$$

$$\overline{Y}_{(pq)} = -g_{(pq),i}\,X^i\,. \tag{14}$$

Let us notice, that the generators $\overline{Y}_k^{(a)}$ annul in the coordinate representation (if we take as X^i vectors of natural basis with zero Lie brackets). On the other hand, the generators $\overline{Y}_{(pq)}$ annul in the tetrad formulation as a consequence of constant values of $g_{(ab)}$.

The weakly conserved field t^i can be written as

$$t^k = N^{ik} X_i + N^{ijk} X_{i;j} \tag{15}$$

and N^{ijk} can be used for the calculation of the superpotential

$$U^{ik} = -N_j{}^{ik} X^j \ . \tag{16}$$

The strongly conserved field S^i is derived from this superpotential

$$S^i = U^{ik}{}_{;k} \tag{17}$$

According to (11), (13), (14)

$$N^{ijk} = \frac{\partial L}{\partial e^{(a)}_{j,k}} e^{(a)i} \ . \tag{18}$$

The derivatives of Lagrange function with respect to the first derivatives of $g_{(ab)}$ do not influence the superpotential as a consequence of scalar character of $g_{(ab)}$. According to (18), (16), (17) and (12) it is possible to determine the conserved currents from the N^{ijk} and G^{ik}.

The energy–momentum complex of gravitational field $t^i_{(a)}$ is formed by four currents related to four independent generating field $X^i_{(a)}$. It fulfills the differential conservation laws

$$t^i_{(a);\,i} = 0$$

resp.

$$\left(\chi t^i_{(a)}\right)_{,\,i} = 0 \tag{19}$$

in vacuo and

$$\left(t^i_{(a)} + T^i_{(a)}\right)_{;\,i} = 0$$

resp.

$$\left[\chi\left(t^i_{(a)} + T^i_{(a)}\right)\right]_{,\,i} = 0 \tag{20}$$

in presence of matter with the energy–momentum tensor T^i_k.

Here, $T^i_{(a)} = T^i_k X^k_{(a)}$; $\chi = \sqrt{-\tilde{g}}\ |e|$ is the volume density.

4. General expression for the superpotential and its specification to the Einstein and Møller case

The calculation of N^{ijk} (18) and U^{ik} (16) is straightforward, but rather tedious as a consequence of complicated explicit dependence of $L = -\frac{1}{2\kappa}\,G$ (15) on its arguments. Finally we obtain

$$N^{ijk} = -\frac{1}{2\kappa}\left[\left(g^{ji}e^{k}_{(a)} - g^{ki}e^{j}_{(a)}\right)e^{(a)s}_{;s} + \right.$$

$$\left. + \left(g^{ji}e^{(a)k} - g^{ki}e^{(a)j}\right)e^{s}_{(a);s} + \left(e^{(a)k;i}e^{j}_{(a)} - e^{(a)j;i}e^{k}_{(a)}\right)\right] \,, \tag{21}$$

$$U^{kl} = -\frac{1}{2\kappa}\,X^{r}\,\left[\left(\delta^{l}_{r}e^{k}_{(a)} - \delta^{k}_{r}e^{l}_{(a)}\right)e^{(a)s}_{;s} + \right.$$

$$\left. + \left(\delta^{l}_{r}e^{(a)k} - \delta^{k}_{r}e^{(a)l}\right)e^{s}_{(a);s} + \left(e^{l}_{(a)}e^{(a)k}_{;r} - e^{k}_{(a)}e^{(a)l}_{;r}\right)\right] \,. \tag{22}$$

In the case of Møller tetrad formulation we put in (22) the generators equal to the tetrad vectors

$$X^{r}_{(a)} = h^{r}_{(a)}\,, \quad g_{(ab)} = \eta_{ab} \tag{23}$$

and using the constantness of $g_{(ab)}$ we immediately obtain

$$U^{kl}_{a} = -\frac{1}{\kappa}h^{r}_{a}\left[\left(\delta^{l}_{r}h^{(b)k} - \delta^{k}_{r}h^{(b)l}\right)h^{k}_{(b);k} + h^{l}_{(b)}h^{(b)k}_{;r}\right] \,, \tag{24}$$

and it is the well–known Møller expression.

For the Einstein formulation we put in (22)

$$X^{r}_{(a)} = \delta^{r}_{a} = e^{r}_{(a)}\,, \quad g_{(ab)} = g_{ab}\,. \tag{25}$$

The calculations are rather longer with regard to the necessity to work with the Christoffel symbols in covariant derivatives. Finally we obtain the expression which can be written in a compact form after transition to the density of the superpotentional

$$\mathcal{U}^{kl}_{i} = \sqrt{-g}\,U^{kl}_{i} \;=\; -\frac{1}{2\kappa}\frac{g_{ir}}{\sqrt{-g}}\left[-g\left(g^{sk}g^{rl} - g^{sl}g^{rk}\right)\right]_{,s}\,. \tag{26}$$

This is the well–known Einstein–Freud superpotential. We have obtained it here directly from the Noether theorems for invariant Lagrangian. It was allowed by the scalar character of our G with respect to the coordinate transformations.

5. Concluding remarks

The presented formulation of conservation laws places the Møller and Einstein solution into a broader class of conserved quantities, dependent on a choice of field of systems of reference (reper $e_i^{(a)}$) which is in general neither holonomic (Einstein) nor orthonormal (Møller). Such system could be most natural for a description of gravitational field in some cases (e.g. in nonisotropic cosmological models).

Traditionally, a "nonlocalizibility" of conserved quantities was seen as a principal source of difficulties connected with the conservation laws in General Relativity. According to our approach, this nonlocalizibility could be identified with a gauge noinivariance of conserved quantities with respect to general nonlinear transformation of repers. But reper determines a system of reference and it is perhaps not so unacceptable to have conserved quantities dependent on the choice of reference system. It is possible that a progress in understanding of conservation laws in General Relativity needs above all a more profound interpretation.

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