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CONFORMAL TRANSFORMATION OF EQUIVALENCE IN QUANTUM COSMOLOGY

STEFAN REUTER

ABSTRACT. The theorem of conformal equivalence points out conditions under which the Einstein Lagrangian plus scalar field can be transformed conformally into a Lagrangian of higher order. Here the conformal equivalence is investigated for the Friedman model if both theories are quantized. Therefore we introduce an intrinsic formalism to quantize the higher order theory in the framework of the Hamiltonian formalism.

Introduction

Conformal transformations of equivalence have been considered in several contexts in cosmology. In particular, the well known conformal equivalence between Einstein's theory plus a minimally coupled scalar field and a fourth order theory of gravity gave reason for discussion. It was used as a starting-point in the argument about the real physical metric: Cotsakis [1] claimed that only Einstein's theory plus scalar field has a physical significance; others [2] argued that the question is not answered still, and both possibilities have to be taken in account equally.
In a slightly different application the conformal tranformation of equivalence was used to consider quantized higher order theories in the picture of Einstein's theory plus scalar field [3]. However Duff [4] gave an example that conformal equivalence between classical systems can break down if these systems are quantized. So it is not clear that it is possible to consider quantized higher order theories in the picture of Einstein's theory plus a minimally coupled scalar field.
In this paper, both the Einstein theory plus a scalar field and the higher order theory are quantized for some special cases (i.e. cosmological models). The resulting Wheeler-de Witt-equations are investigated with respect to their conformal equivalence. It is possible to specify conditions which allow the classical conformal transformation to work for the quantized systems.

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In order to derive a Wheeler-de Witt-equation for higher order gravity it is necessary to introduce new variables. They render possible the application of the Hamiltonian formalism. Here we introduce an intrinsic definition of one new variable, intrinsic in so far, that the definition of the new variable turns out to be equivalent to one of the canonical equations. Conditions for such a definition of new variables have been given by Schmidt [5] in a more general framework.
The paper begins with a short summary of the theorem of conformal equivalence as it was given by Bicknell in 1974 [6], [7].

1. Bicknell's Theorem of Conformal Equivalence

Bicknell's starting-point was work done by Rainich (1925) [8] and later Misner and Wheeler (1957) [9], who showed that the Einstein-Maxwell-equation could be expressed completely in geometrical terms. This can be done for any non-gravitional field with a symmetric energy-momentum tensor. The so-called Rainich geometrization leads to an geometrization of physics in general and was looked upon as an alternative for an unified field equation.
However there is one defect of Rainich geometrization: it is not always possible to state an action integral for every given field equation. This implies that the consistency of the field equation is not secured in general. Nevertheless Bicknell succeeded in giving an action integral for the neutral meson field, which Rainich geometrization has been done by Kuchar [10] (1963).

The Lagrangian

Bicknell considered the Langragian $L = f(R)$, with R being the curvature scalar derived by the metric g_{ij} and $f(R)$ an arbitrary C^3-function in R and $f'f'' \neq 0$, where we define $f' := \frac{df}{dR}$ and $f'' := \frac{d^2 f}{dR^2}$. Using the fourth order field equation according to Buchdahl [11] (here we give it in a compressed form):

$$(1) \qquad R_{ij} f'(R) - \frac{1}{2} f(R) g_{ij} - f'(R)_{;ij} + g_{ij} \Box f'(R) = 0 \qquad .$$

performing the following transformation for $f' \neq 0$

$$\chi = \sqrt{\frac{3}{2}} \ln(\epsilon f') \qquad \text{with} \qquad \epsilon = sgn(f') \qquad ,$$

and introducing the well known conformal transformations for the curvature scalar and the Ricci tensor and the condition for the scalar field χ

$$\tilde{\Box}\chi = p^2 \chi \quad \text{with} \quad p \in R \qquad \text{, i.e. the Klein-Gordon equation,}$$

Bicknell got the conditions how the higher order Lagrangian transforms conformally. Here we use a slightly different conformal transformation to ease the comparison to the recent literature [3]. So the different parts of the Einstein Lagrangian plus scalar field Φ can be expressed in terms of the higher order Lagrangian.

$$\begin{array}{rrcl} \text{metric:} & \tilde{g}_{ij} & = & -2f' g_{ij} \\ \text{field term:} & \Phi & = & \sqrt{\frac{3}{2}} \ln(-2f') \\ \text{potential term:} & V & = & \frac{1}{4f'^2}(Rf' - f) \end{array} \tag{2}$$

(To better distinguish between Einstein's theory and the higher order theory expressions which occur in both theories are denoted in Einstein's theory by a tilde: $\tilde{}$.) Bicknell restricted his consideration of the potential term to a constant, i.e. a cosmological term.

We now try to use the Bicknell theorem in order to compare the behaviour of the corresponding Wheeler-de Witt equations of Einstein's theory and a fourth order theorie under a conformal transformation of equivalence.

2. Conformal Equivalence for the Quantized Friedman model

The lack of a general theory of quantum gravity forces us to constrain our considerations to special, highly symmetric models, i.e., in terms of superspace we restrict to minisuperspaces.

Because of the uncertainty principle the Wheeler-de Witt equation is not determined uniquely just by the classical Hamiltonian. This is called the factor-ordering problem. Here we follow the proposal by Hawking and Page [12] to select the Wheeler-de Witt-equation that is covariant with respect to the metric in minisuperspace.

In addition, we introduce one more restriction as follows: we adopt a time gauge by using conformal time. Although time gauge and variation procedure can not be exchanged arbitrarily it is allways possible to reach this special time gauge [13].

The Wheeler-de Witt equation for $L_S = -\frac{\tilde{R}}{2} + \frac{1}{2}g^{ij}\Phi_{,i}\Phi_{,j} - V(\Phi)$

First we generalize some previous results [13] for the Friedman model. Under the above conditions the metric can obtain the following form:

$$ds^2 = e^{2\alpha(t)}\left[dt^2 - \frac{dr^2}{1-kr^2} - r^2(d\theta^2 + sin^2\theta d\psi^2)\right] \quad , k \in \{-1, 0, 1\} . \tag{3}$$

We assume that the considered scalar field Φ is just time dependent. $\tilde{g}^{ij}\Phi_{,i}\Phi_{,j} = e^{-2\alpha}\dot{\Phi}^2$ (The dot denotes the derivative with respect to time). With the above defined Lagrange scalar, L_S, we derive the Lagrangian, after adding a divergence, via performing an integration over the spatial coordinates. Then we introduce the generalized coordinates $\dot{\alpha}$ and $\dot{\Phi}$. Thus the canonical momenta, $\tilde{\pi}_A := \frac{\partial \tilde{L}}{\partial \frac{d\tilde{q}^A}{dt}}$, and the metric of minisuperspace yield:

$$\tilde{\pi}_1 = -6\dot{\alpha}e^{2\alpha} \quad , \quad \tilde{\pi}_2 = \dot{\Phi}e^{2\alpha} \quad , \quad \tilde{g}_{AB} = \frac{\partial^2 \tilde{L}}{\partial \dot{q}^A \partial \dot{q}^B} = \begin{pmatrix} -6e^{2\alpha} & 0 \\ 0 & e^{2\alpha} \end{pmatrix} .$$

(The index in minisuperspace is denoted by a capital latin letter, here they adopt the value 1 or 2.)
After performing the Legendre transformation $H = \pi_A \dot{q}^A - L$, the Hamiltonian can be expressed in terms of the canonical momenta.

$$\tilde{H} = -\frac{1}{12}\tilde{\pi}_1^2 e^{-2\alpha} + \frac{1}{2}\tilde{\pi}_2^2 e^{-2\alpha} - 3ke^{2\alpha} + V(\Phi)e^{4\alpha} \qquad .$$

If we introduce the Hamiltonian constraint $\tilde{H} = 0$ and perform the transformation from the canonical momenta to differential operators $\tilde{\pi}_A \to \frac{\hbar}{i}\tilde{\nabla}_A$, we get the Wheeler-de Witt equation. Here $\tilde{\nabla}_A$ denotes the covariant derivative with respect to $\tilde{q}^A$, which results from the metric of minisuperspace, which is covariantly constant with respect to minisuperspace. Thus $\tilde{g}_{AB}$ and $\tilde{\nabla}_A$ can be exchanged arbitrarily, the factor-ordering problem does not occur at all. So the transition to the covariant derivatives in minisuperspace gives:

$$\hat{H}\psi = \left(-\frac{\hbar^2}{2}\tilde{\Box}_{MS} - 3ke^{2\alpha} + V(\Phi)e^{4\alpha}\right)\psi \qquad ,$$

where the index MS symbolizes the derivative with respect to coordinates of minisuperspace. In our case the d'Alembertian operating on the scalar ψ can be expressed easily in partial derivatives with respect to α and Φ. The factor-ordering problem has been solved in the postulated way, i.e. that the Wheeler-de Witt equation has to be covariant with respect to the metric of minisuperspace. Finally we get:

$$0 = \hbar^2\left(\psi_{,\alpha\alpha} - 6\psi_{,\Phi\Phi}\right) + \left(12V(\Phi)e^{6\alpha} - 36ke^{4\alpha}\right)\psi \qquad . \tag{4}$$

In this form the Wheeler-de Witt equation has been given by Zeh in (1989) [14] and analogously by Gibbons and Grishuk (1989) [15]. In the following section we will show how this equation can be transformed conformally into a Wheeler-de Witt equation of a fourth order theory.

2.1. The Wheeler-de Witt equation for $L_S = f(R)$

In order to derive a Wheeler-de Witt equation for a higher order theory for an arbitrary $f(R)$ we have to reduce the order of the system. Usually this is done by defining a new coordinate and use the formalism for constrained Hamiltonian systems. Here we want to introduce an intrinsic definition of this new variable, i.e. the definition is already given by one canonical equation. Schmidt [5] has considered such definitions in a more general view. Again we constrain ourselves to the special case of the Friedman model for $L_S = f(R)$. Now we choose the metric in a slightly different way (compare eq. (3)),

$$ds^2 = a(t)^2\left(dt^2 - \frac{dr^2}{1-kr^2} - r^2(d\theta^2 + sin^2\theta d\psi^2)\right) \qquad \text{with} \tag{5}$$

a being scalar factor of the metric. Thus the curvature scalar is expressed by: $R = -\frac{6}{a^3}(\ddot{a} + ka)$. The results in [13] suggest following definition of the new variable: $b := -f'(R)a$. The inverse function of $f'(R)$, g is defined by

$$g(-\frac{b}{a}) = R \qquad \text{, thus} \quad f'(g(-\frac{b}{a})) = -\frac{b}{a} \quad .$$

It is determined uniquely in an interval I with $R \in I$, in which $f'f'' \neq 0$ is valid. Using the expression for the curvature scalar we get for the Friedman model:

$$\ddot{a} = -\frac{1}{6}\left(g(-\frac{b}{a})a^3 + 6ka\right) \quad . \tag{6}$$

The Divergence

While the transition from the Lagrange scalar to the Lagrangian it is possible to add a divergence, which changes while integration over the space-time into a surface integral. The resulting field equations are equivalent. Trying to generalize the results of [13], we assume for this divergence: $L^{div} = \ddot{a}h(a,b) + \dot{a}^2 h_a + \dot{a}\dot{b}h_b$, where h_a and h_b are defined by $h_a := \frac{\partial h(a,b)}{\partial a}$ and $h_b = \frac{\partial h(a,b)}{\partial b}$. The second derivatives are denoted analogously by h_{aa}, h_{ab} and h_{bb}.
The integration over the Lagrangian density (the case $k = -1$ has to be excluded because the hyperbolic space-time is not compact) yields:

$$L = f(g(-\frac{b}{a}))a^4 - \frac{1}{6}\left(a^3 g(-\frac{b}{a}) + 6ka\right)h + \dot{a}^2 h_a + \dot{a}\dot{b}h_b \quad . \tag{7}$$

We choose a and b as generalized coordinates and get for the canonical momenta and the metric of minisuperspace g_{AB}:

$$\pi_1 = (2\dot{a}h_a + \dot{b}h_b)\,, \quad \pi_2 = \dot{a}h_b\,, \quad g_{AB} = \begin{pmatrix} 2h_a & h_b \\ h_b & 0 \end{pmatrix}\,,\ A,B \in \{1,2\}\ . \tag{8}$$

As in the previous section the Hamiltonian can be derived from the Lagrangian by the Legendre transformation. Then we introduce the momenta to the Hamiltonian and get finally:

$$H = \frac{1}{h_b}\pi_1\pi_2 - \frac{h_a}{h_b^2}\pi_2^2 - f(g(-\frac{b}{a}))a^4 + \frac{1}{6}\left(g(-\frac{b}{a})a^3 + 6ka\right)h \quad . \tag{9}$$

The Canonical Equations

Now let us consider the canonical equations. Those, resulting from $\frac{\partial H}{\partial \pi_A} = \dot{q}^A$, give us a consistency check. Of the two other equations we first discuss $\frac{\partial H}{\partial b} = -\dot{\pi}_2$. Using eqs. (8), (9), the definition of f and g the left side of this equation consists of several terms: one is an expression in second derivatives in h, one term contains h_b and $\ddot{a}$ and the remaining terms contain g'. The right side gives the same expressions in second order of h and the same expression in h_b. Finally we get a unique solution for h: $h = -6b$. It is surprising that the solution for h is so simple. One should have expected a second order differential equation defining h. This solution will be used in the following. So we get for the divergence: $L^{div} = -\frac{d}{dt}(6\dot{a}b)$ determined uniquely in the framework of our ansatz. The expression for the Hamiltonian can be simplified to:

$$H = -\frac{1}{6}\pi_1\pi_2 - f(g(-\frac{b}{a}))a^4 - a^3bg(-\frac{b}{a}) - 6kab \qquad . \tag{10}$$

Now we want to show that the other canonical equation, $\frac{\partial H}{\partial q^1} = -\dot{\pi}_1$, is equivalent to the trace of the fourth order field equation. Using the definition of g the canonical equation gives:

$$6\ddot{b} = -4a^3 f(g(-\frac{b}{a})) - 3a^2bg(-\frac{b}{a}) - 6kb \qquad . \tag{11}$$

We compare this result with the trace of eq. (1):

$$Rf'(R) - 2f(R) + 3\,\Box\, f'(R) = 0 \qquad .$$

Using the definition of $f(g(-\frac{b}{a})) = -\frac{b}{a}$ the trace equation simplifies to:

$$-\frac{b}{a}g(-\frac{b}{a}) - f(g(-\frac{b}{a})) - 3\,\Box\,\frac{b}{a} = 0 \qquad . \tag{12}$$

First we calculate the last term, where we consider just time dependent a and b:

$$\Box\frac{b}{a} = \frac{\ddot{b}}{a^3} + \frac{b}{6a}\left[g(-\frac{b}{a}) + \frac{kb}{a^3}\right] \qquad ,$$

where we inserted eq. (6), the relation between $\ddot{a}$ and g.

This result we introduce into eq. (12), multiply with $2a^3$ and pose $\ddot{b}$ on the right side of the equation. We finally obtain the above calculated canonical equation. So we were able to show, that one canonical equation gives the definition of the variable b and the other is equivalent to the trace of the fourth order field equation for arbitrary $f(R)$, while $H = 0$ is equivalent to the 00-component of the field equation.

The Wheeler-de Witt equation

With the Hamiltonian constraint as the 00-component of the fourth order field equation we can now start to derive the Wheeler-de Witt equation for $L_S = f(R)$. Inserting the result for h in eq. (10) we get following Lagrangian:

$$L = f(g(-\frac{b}{a}))a^4 + a^3bg(-\frac{b}{a}) + 6kab - 6\dot{a}\dot{b} \qquad .$$

The metric of minisuperspace becomes:

$$g_{AB} = \frac{\partial^2 L}{\partial\dot{q}^A\partial\dot{q}^B} = \begin{pmatrix} 0 & -6 \\ -6 & 0 \end{pmatrix}$$

We put $L_S = f(R)$ and obtain for the Lagrangian density: $\mathcal{L}^* = f(R)\sqrt{-g}$. In order to transform this system of higher order we introduce the new variable b

in the above discussed way. The Hamiltonian in eq. (10) now can be expressed in terms of the metric of minisuperspace:

$$H = \frac{1}{2} g^{AB} \pi_A \pi_B - f(g(-\frac{b}{a}) - a^3 bg(-\frac{b}{a}) - 6kab.$$

Introducing the Hamiltonian constraint from the 00-component of the field equation and performing the transition $\pi_a \to \frac{\hbar}{i} \nabla_A$ we now can name the Wheeler-de Witt equation for a Lagrangian $f(R)$ in the Friedman model:

$$\left(-\frac{\hbar^2}{2} \Box_{MS} - f(g(-\frac{b}{a}))a^4 - a^3 bg(-\frac{b}{a}) - 6kab \right) \psi = 0 \qquad .$$

The metric of minisuperspace is constant, so the covariant derivative can be expressed by the partial derivative. Finally we can write for the Wheeler-de Witt equation:

$$\hbar^2 \psi_{,12} - 6 \left[f(g(-\frac{b}{a}))a^4 + a^3 bg(-\frac{b}{a}) + 6kab \right] \psi = 0 \qquad . \tag{13}$$

The Conformal Transformation

Using the classical conformal relation (eq.(2)) we now start to transform eq. (12) into eq. (4). To ease this procedure we introduce a new variable $\beta := e^{\alpha}$, transform equation (6) and multiply with $\frac{1}{2\beta^2}$:

$$0 = \frac{\hbar^2}{2} (\psi_{,\beta\beta} + \psi_{,\beta} \frac{1}{\beta} - \frac{6}{\beta^2} \psi_{,\Phi\Phi}) + 6(V(\Phi)\beta^4 - 3k\beta^2)\psi \qquad .$$

$V(\Phi)$ transforms according to eq. (2). We insert the relations for f and g:

$$\beta^4 V(\Phi) - 3k\beta^2 = -a^3 bg(-\frac{b}{a}) - a^4 f(g(-\frac{b}{a})) - 6kab \qquad .$$

We may set $\beta = \sqrt{2ab}$ because this is exactly the conformal transformation of the scale factor (comp. eq. (3)). So we have already seen that the conformal transformation performs well for the potential terms of the corresponding Wheeler-de Witt equations. Now we start from eq. (13) and calculate the partial derivatives of the coordinates Φ and b with respect to the coordinates a and b: Now we can calculate the partial derivatives:

$$\frac{\partial \beta}{\partial a} = \sqrt{\frac{b}{2a}} \, , \; \frac{\partial \Phi}{\partial a} = -\sqrt{\frac{3}{2}} \frac{1}{a} \, , \; \frac{\partial \beta}{\partial b} = \sqrt{\frac{a}{2b}} \, , \quad \frac{\partial \Phi}{\partial b} = \sqrt{\frac{3}{2}} \frac{1}{b} \, , \; \frac{\partial^2 \beta}{\partial a \partial b} = \frac{1}{2} \sqrt{\frac{1}{2ab}} \quad . \tag{14}$$

Hence it is possible to transcribe $\psi_{,ab}$ and use: $\frac{\partial^2 \Phi}{\partial a \partial b} = 0$, thus we get:

$$\psi_{,ab} = \frac{1}{2} \left(\frac{1}{\beta} \psi_{,\beta} + \psi_{,\beta\beta} - \frac{6}{\beta^2} \psi_{,\Phi\Phi} \right) \qquad ,$$

where the mixed terms in Φ and β cancel each other. Hence we are able to transform the dynamical part of the Wheeler-de Witt equation as well and have shown, that at least for the Friedman model the conformal transformation is valid for the quantized case as well.

Conclusion and Outlook

We were able to state conditions under which the conformal transformation of equivalence works for the quantized systems as well. The postulation of covariance with respect to minisuperspace of the Wheeler-de Witt equation gets support. If the conformal equivalence of the classical Lagrangian systems has a deeper physical significance than the pure mathematical one, then we have a situation in which the minisuperspace covariant Wheeler-de Witt equation has a special position in comparism to other solutions of the factor-ordering problem.
In order to quantize the higher order theory we had to introduce a special time gauge and an intrinsic definition of the additional variable. This method is unusual, but as the work by Schmidt has shown that this intrinsic definition can work for more general situations. The postulation of the intrinsic definition of the introduced variable can be looked upon as a constraint to the system. We also used the field equation, with the Hamiltonian constraint as 00-component as a starting point. So it would be satisfactory, if it was possible to derive this intrinsic constraints from the usual formalism, i.e. to consider a constrained Hamiltonian system, where the definition of an introduced variable is a constraint to the system. Also the theorem of conformal transformation is more general in the quantized case. For some special cases it works for anisotropic but homogenous cosmological models. It was possible to prove the quantized theorem for cosmological models of Bianchi type I and II. The structure of the proof we have given here gives raise to the hope, that the theorem can be extended to all homogenous cosmological models. In such a model it is possible to find a metric with a scalar factor and other coordinates, which express the anisotropy (the so-called Misner-parametrization). Under this conditions it could be possible to transpose the proof for anisotropic but homogenous cosmological models as well.
At least the quantized Bicknell theorem for the Friedman model can be used to find solutions for the Wheeler-de Witt equation derived from the Einstein Lagrangian. So it can contribute to the discussion how typical the inflationary phase may be in the set of all universes. This question is one of the primary problems in Quantum Cosmology.

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Universität Potsdam, Institut für Mathematik, Projektgruppe Kosmologie
Am Neuen Palais 10, D-14415 Potsdam, PF 601553, Germany

E-mail: tirza@rz.uni-potsdam.de