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THE GEOMETRY OF MIXED FIRST AND SECOND-ORDER DIFFERENTIAL EQUATIONS WITH APPLICATIONS TO NON-HOLONOMIC MECHANICS

W SARLET

ABSTRACT. A geometrical framework is presented for a class of dynamical systems, which are modelled by a mixed system of first and second-order ordinary differential equations. The starting point for the model is a bundle $\pi : E \to M$, where both E and M are fibred over $I\!R$. We show that there are two connections associated to systems of this kind: the first one comes from the 'constraint equations' (the first-order equations); the second one is related to the full system on the constraint manifold. Among other things, we discuss the concepts of symmetry and adjoint symmetry for such systems and identify for that purpose an appropriate notion of 'dynamical covariant derivative' and 'Jacobi endomorphism'. The curvature tensors of the two connections play an important role in defining these concepts. The present approach in particular is relevant for modelling certain mechanical systems with non-holonomic constraints.

1. Introduction

The objects of study in this paper are systems of differential equations of the following type:

$$\begin{aligned} \ddot{q}^\alpha &= f^\alpha(t, q^A, \dot{q}^\beta), & \alpha &= 1, \dots, k \\ \dot{q}^a &= g^a(t, q^A, \dot{q}^\beta), & a &= 1, \dots, m, \end{aligned}$$

where the coordinates q^A collectively denote the (q^α, q^a), i.e. the index A will run from 1 to $n = k + m$.

There are at least two different situations in which equations of this type make their appearance in applications. They could for example originate from a larger set of second-order equations after a process of reduction of order via a solvable

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algebra of symmetries. An entirely different origin for such equations is the theory of mechanical systems with non-holonomic constraints, which most of the time will be linear, say of the form

$$g^a(t, q^A, \dot{q}^\beta) = B^a_\alpha(t, q^A)\,\dot{q}^\alpha + B^a(t, q^A).$$

By way of example, let us briefly describe under what circumstances Lagrangian systems with non-holonomic constraints ultimately give rise to equations of motion of the above type. Assume that a mechanical system, originally governed by a (free) Lagrangian $L(t, q^A, \dot{q}^A)$, $A = 1, \ldots, n = k + m$, is being subject to (linear) non-holonomic constraints of the explicit form

$$\dot{q}^a = B^a_\alpha(t, q^A)\dot{q}^\alpha + B^a(t, q^A), \qquad a = 1, \ldots, m.$$

In the classical literature (see e.g. [7]), these would probably be called generalized Čaplygin constraints. Following the standard procedure of introducing Lagrange multipliers to cope with these constraints in Hamilton's principle, one arrives at equations of motion from which it is easy to eliminate the multipliers (see [8, 1] for more details). The resulting equations become

$$\frac{d}{dt}\left(\frac{\partial \overline{L}}{\partial \dot{q}^\alpha}\right) = X_\alpha(\overline{L}) + C^a_\alpha\,\frac{\partial L}{\partial \dot{q}^a},$$

where

$$\begin{aligned}
\overline{L}(t, q^A, \dot{q}^\alpha) &\equiv L(t, q^A, \dot{q}^\alpha, B^a_\beta \dot{q}^\beta + B^a) \\
X_\alpha &= \frac{\partial}{\partial q^\alpha} + B^a_\alpha\,\frac{\partial}{\partial q^a} \\
C^a_\alpha &= \dot{B}^a_\alpha - X_\alpha(B^a_\beta \dot{q}^\beta + B^a).
\end{aligned}$$

Assuming regularity for the reduced Lagrangian $\overline{L}$, in the sense that we have $\det\left(\partial^2 \overline{L}/\partial \dot{q}^\alpha \partial \dot{q}^\beta\right) \neq 0$, these second-order equations can be written in normal form and thus, together with the constraint equations, produce a subclass of the type of mixed systems we want to investigate here.

For the ease of terminology, whatever the origin of our mixed system, we will refer to the first-order equations as the constraint equations. In our geometrical model, they will define a submanifold of an appropriate jet bundle (the 'constraint submanifold'). One of the primary tasks then is to give a coordinate free meaning to second-order equations living on such a constraint submanifold and to study the geometrical structures which are naturally associated to dynamical systems of this kind. Before we get to these issues, however, which will constitute the major portion of this paper in Sections 3 to 5, we will first discuss in the next section the geometry which comes for free with the constraint equations themselves.

The work which is reported here is being developed in collaboration with F. Cantrijn and D.J. Saunders. It concerns a generalization of the case of linear constraints, as described in [8]. Other aspects of this theory, in particular the way one can design a direct geometrical construction of the reduced dynamics of non-holonomic systems, without the intermediate process of introducing multipliers, have been highlighted in [9] and will therefore only briefly be referred to in the last section.

2. Connections associated to the constraint equations

The nature of the mixed systems of differential equations under investigation, where one can clearly make a distinction between two types of dependent variables while the equations further explicitly depend on the independent time coordinate, seems to suggest that an appropriate geometrical set-up should start from the structure of a bundle $\pi : E \to M$, with both E and M fibred over $I\!R$:

$$
\begin{array}{ccccc}
(t, q^\alpha, q^a) & E & \xrightarrow{\ \pi\ } & M & (t, q^\alpha) \\
 & \Big\downarrow \tau_1 & & \tau_0 \Big\downarrow & \\
 & I\!R & \xrightarrow{\ id\ } & I\!R &
\end{array}
$$

It is of some interest to continue for a moment talking about the special case of linear constraints, because these more clearly bring some further structure into the picture, which in turn justifies the query for similar constructions in the general case.

Looking at the two fibrations over $I\!R$, it is natural to introduce the jet spaces $J^1\tau_0$ and $J^1\tau_1$ and to observe that the latter is also fibred over the pull-back bundle $\pi^* J^1\tau_0$. Linear constraint equations $\dot q^a = B^a_\alpha(t, q^A)\dot q^\alpha + B^a(t, q^A)$ clearly define a section σ of this fibration, but one which actually represents a connection on $\pi : E \to M$, i.e. corresponds also to a section $\tilde\sigma$ $(q^a_t = B^a(t, q^A),\ q^a_\alpha = B^a_\alpha(t, q^A))$ of $J^1\pi \to E$. As has been shown in [8], the connection $\tilde\sigma$ further has a natural lift to a connection $\tilde\sigma_1$ on the bundle $\pi_1 : \pi^* J^1\tau_0 \to J^1\tau_0$.

In the more general case of non-linear constraints $\dot q^a = g^a(t, q^A, \dot q^\beta)$, we are looking at a general section of $J^1\tau_1$ over $\pi^* J^1\tau_0$, which is not related to a connection on π. It so happens, however, that a general section σ still defines a connection $\tilde\sigma_1$ on π_1. One possible way to exhibit this connection $\tilde\sigma_1$ consists in specifying the following horizontal lift rules from vector fields on $J^1\tau_0$ to vector fields on $\pi^* J^1\tau_0$:

$$
\begin{aligned}
\frac{\partial}{\partial t} \quad &\longmapsto \quad \frac{\partial}{\partial t} + \left(g^a - \dot q^\alpha \frac{\partial g^a}{\partial \dot q^\alpha} \right) \frac{\partial}{\partial q^a} \\
\frac{\partial}{\partial q^\alpha} \quad &\longmapsto \quad \frac{\partial}{\partial q^\alpha} + \frac{\partial g^a}{\partial \dot q^\alpha} \frac{\partial}{\partial q^a},
\end{aligned}
$$

$$\frac{\partial}{\partial \dot{q}^\alpha} \quad \longmapsto \quad \frac{\partial}{\partial \dot{q}^\alpha}.$$

Solution curves of our mixed system will be curves $q^A(t)$ in E satisfying the constraints, i.e. whose prolongation to $J^1\tau_1$ lies in the image of σ. So, we are led to define the *constraint manifold* $J^1_\sigma = \sigma(\pi^* J^1\tau_0)$ and the problem of modelling the complete dynamics is now a matter of defining an appropriate vector field on J^1_σ. For that purpose, we will of course make use of intrinsic structures living on the constraint manifold. One which we can collect from the present discussion is a type (1,1) tensor field N: the 'vertical projection operator' associated to the connection $\tilde{\sigma}_1$ (by construction this is a tensor field on $\pi^* J^1\tau_0$ and it carries over to J^1_σ via the diffeomorphism σ). In coordinates $(t, q^A, \dot{q}^\alpha)$ on J^1_σ, N is of the form

$$N = \eta^a \otimes \frac{\partial}{\partial q^a}, \qquad \eta^a = dq^a - B^a_\alpha dq^\alpha - B^a dt,$$

where, for the case of general constraints $\dot{q}^a = g^a(t, q^A, \dot{q}^\beta)$, the functions B^a_α and B^a are defined by

$$B^a_\alpha(t, q^A, \dot{q}^\beta) = \frac{\partial g^a}{\partial \dot{q}^\alpha}, \qquad B^a(t, q^A, \dot{q}^\beta) = g^a - \dot{q}^\alpha B^a_\alpha.$$

The 1-forms η^a can be called the *constraint forms*.

3. Second-order equations on the constraint submanifold

There is another important type (1,1) tensor field living on J^1_σ, finding its roots in the jet bundle structure of the space. Indeed, the canonical 'vertical endomorphism' on $J^1\tau_0$,

$$S = \theta^\alpha \otimes \frac{\partial}{\partial \dot{q}^\alpha}, \qquad \theta^\alpha = dq^\alpha - \dot{q}^\alpha dt,$$

is well defined also on $\pi^* J^1\tau_0$ and via the diffeomorphism σ subsequently on J^1_σ.

A second-order differential equation field (SODE) $\Gamma \in \mathcal{X}(J^1_\sigma)$ is defined by the following characterizing properties:

$$\langle \Gamma, dt \rangle = 1, \quad \langle \Gamma, \theta^\alpha \rangle = 0, \quad \langle \Gamma, \eta^a \rangle = 0.$$

In coordinates, Γ is of the form

$$\Gamma = \frac{\partial}{\partial t} + \dot{q}^\alpha \frac{\partial}{\partial q^\alpha} + g^a(t, q^A, \dot{q}^\beta) \frac{\partial}{\partial q^a} + f^\alpha(t, q^A, \dot{q}^\beta) \frac{\partial}{\partial \dot{q}^\alpha},$$

and thus truly models the type of mixed differential equations of interest.

As in the standard theory of second-order differential equations, Γ comes with its own connection on the bundle $\rho : J^1_\sigma \to E$. To discover this connection, it suffices for example to calculate that

$$(\mathcal{L}_\Gamma S)^2 = I - \Gamma \otimes dt - N,$$

from which it follows that

$$\begin{aligned} P_H &= \frac{1}{2}(I - \mathcal{L}_\Gamma S + \Gamma \otimes dt + N), \\ P_V &= \frac{1}{2}(I + \mathcal{L}_\Gamma S - \Gamma \otimes dt - N), \end{aligned}$$

are two complementary projection operators. The associated 'horizontal lift' operation from $\mathcal{X}(E)$ to $\mathcal{X}(J^1_\sigma)$ is given by

$$\begin{aligned} \frac{\partial}{\partial t} &\longmapsto \frac{\partial}{\partial t} + (f^\alpha + \dot q^\beta \Gamma^\alpha_\beta)\frac{\partial}{\partial \dot q^\alpha} \\ \frac{\partial}{\partial q^\alpha} &\longmapsto \frac{\partial}{\partial q^\alpha} - \Gamma^\beta_\alpha \frac{\partial}{\partial \dot q^\beta}, \quad \text{with} \quad \Gamma^\beta_\alpha = -\frac{1}{2}\frac{\partial f^\beta}{\partial \dot q^\alpha} \\ \frac{\partial}{\partial q^a} &\longmapsto \frac{\partial}{\partial q^a}. \end{aligned}$$

4. Digression: calculus along $\rho : J^1_\sigma \to E$

From earlier work related to unconstrained second-order systems (see [4, 5, 11]), we know the importance of operations (tensorial operations as well as derivations) which are essentially based on calculations involving tensor fields along the projection onto the configuration space (rather than corresponding objects living on the full space where the dynamical system is defined). Motivated by this background, we will briefly sketch some elements of the calculus along the projection ρ of the constraint submanifold onto the configuration manifold E. Our guideline here will be to limit ourselves to those elements which are relevant for a discussion of symmetries and adjoint symmetries of the given system (see next section).

Let $\mathcal{X}(\rho)$ denote the $C^\infty(J^1_\sigma)$ module of vector fields along ρ (maps from J^1_σ to TE). There exists a canonical vector field along ρ, which in coordinates reads:

$$\mathbf{T} = \frac{\partial}{\partial t} + \dot q^\alpha \frac{\partial}{\partial q^\alpha} + g^a \frac{\partial}{\partial q^a}.$$

Note in passing that the horizontal lift associated to the SODE connection, as described at the end of the previous section, extends naturally to $\mathcal{X}(\rho)$ and that, in particular, we have

$$\mathbf{T}^H = \Gamma.$$

In addition, there is a canonical 'vertical lift' operation from $\mathcal{X}(\rho)$ to $\mathcal{X}(J^1_\sigma)$ (essentially determined by the action of S), for which we have

$$\mathbf{T}^V = 0, \quad {X_\alpha}^V = \frac{\partial}{\partial \dot q^\alpha}, \quad \frac{\partial}{\partial q^a}^V = 0.$$

Every $Z \in \mathcal{X}(J^1_\sigma)$ has a unique decomposition in the form

$$Z = X^H + \overline{Y}^V,$$

with $X \in \mathcal{X}(\rho)$ and $\overline{Y} \in \overline{\mathcal{X}}(\rho) \subset \mathcal{X}(\rho)$, where the submodule $\overline{\mathcal{X}}(\rho)$ is defined by

$$\overline{\mathcal{X}}(\rho) = \{\overline{X} \in \mathcal{X}(\rho) |\ \langle \overline{X}, dt \rangle = 0,\ \langle \overline{X}, \eta^a \rangle = 0\}.$$

$\overline{\mathcal{X}}(\rho)$ is locally spanned by

$$X_\beta = \frac{\partial}{\partial q^\beta} + B^a_\beta \frac{\partial}{\partial q^a}.$$

Note further that X itself can be decomposed as

$$X = \langle X, dt \rangle \mathbf{T} + \overline{X} + \widetilde{X},$$

with $\widetilde{X} \in \widetilde{\mathcal{X}}(\rho)$, which is the submodule spanned by the $\partial/\partial q^a$.

Remark: similar decompositions can be identified for forms on J^1_σ and, coming back to the argumentation at the beginning of this section, the interest of the 'calculus along ρ' stems from the fact that many of the geometrical quantities occurring in applications are expected to have their horizontal and vertical part coming essentially from the same object along the projection. The most economical way to describe the application in question in a coordinate free way then should come from a theory directly related to these objects.

We will now define two important type (1,1) tensor fields along ρ which, roughly speaking, come from the 'time-component' of the curvature tensors of both the constraint and the SODE connection.

The curvature of the connection $\tilde{\sigma}_1$ can be defined e.g. via the Nijenhuis tensor $\mathcal{N}_N$ of the vertical projector N, and can be regarded as a vector valued 2-form on J^1_σ. Contraction with Γ gives rise to the following tensor field:

$$\Psi = i_\Gamma \mathcal{N}_N \ = \ C^a_\beta\, \theta^\beta \otimes \frac{\partial}{\partial q^a},$$

where $C^a_\beta \ = \ \Gamma(B^a_\beta) - X_\beta(g^a)$. In principle, Ψ is a tensor field on J^1_σ, but its coordinate expression reveals that it can equally be regarded as a tensor field along ρ.

For the SODE connection, one can view the curvature R directly as a vector valued 2-form along ρ, determined by the vertical part in the decomposition of the bracket of two horizontal lifts (of vector fields along ρ):

$$[X^H, Y^H] = (\cdots)^H + (R(X,Y))^V.$$

Define then:

$$\Phi = i_{\mathbf{T}} R = \Phi^\beta_\alpha\, \theta^\alpha \otimes X_\beta - \frac{\partial f^\beta}{\partial q^a}\, \eta^a \otimes X_\beta$$

where $\Phi^\beta_\alpha = -X_\alpha(f^\beta) - \Gamma^\gamma_\alpha \Gamma^\beta_\gamma - \Gamma(\Gamma^\beta_\alpha)$.

The final ingredient we need for the next section is a degree zero derivation ∇ of tensor fields along ρ, called the *dynamical covariant derivative*. In the context of unconstrained second-order equations, the dynamical covariant derivative has amply shown its relevance in applications such as those reported in [6, 2, 3]. A rather instructive route to its definition comes from playing with the decomposition of vector fields on the full space again, this time in particular the Lie derivative with respect to Γ of a horizontal and vertical lift. In the present context of mixed first and second-order equations modelled by a SODE Γ on the constraint manifold J^1_σ, one can verify that $\forall X \in \mathcal{X}(\rho)$,

$$\begin{aligned}\mathcal{L}_\Gamma X^V &= -\overline{X}^H + (\cdots)^V \\ \mathcal{L}_\Gamma X^H &= (\cdots)^H + \Psi(X)^H + \Phi(X)^V\end{aligned}$$

The identified parts depend linearly on X, the missing terms on the other hand involve a derivation. Together, they determine ∇X, provided we define $\nabla F = \Gamma(F)$ on functions F and 'normalize' the $\mathbf{T}$-component by the requirement $i_{\nabla X}\, dt = \Gamma\big(i_X\, dt\big)$.

With respect to the local basis

$$\mathbf{T}, \quad X_\alpha = \frac{\partial}{\partial q^\alpha} + B^a_\alpha \frac{\partial}{\partial q^a}, \quad \frac{\partial}{\partial q^a}$$

for $\mathcal{X}(\rho)$, ∇ is determined by

$$\nabla \mathbf{T} = 0, \quad \nabla X_\alpha = \Gamma^\beta_\alpha\, X_\beta, \quad \nabla \frac{\partial}{\partial q^a} = -\frac{\partial g^b}{\partial q^a}\frac{\partial}{\partial q^b}.$$

The dual basis of 1-forms along ρ is:

$$dt, \quad \theta^\alpha = dq^\alpha - \dot{q}^\alpha\, dt, \quad \eta^a = dq^a - B^a_\alpha dq^\alpha - B^a\, dt,$$

and the dual action of ∇ on forms then is given by

$$\nabla(dt) = 0, \quad \nabla\theta^\alpha = -\Gamma^\alpha_\beta\, \theta^\beta, \quad \nabla\eta^a = \frac{\partial g^a}{\partial q^b}\, \eta^b.$$

5. Symmetries and adjoint symmetries

Recall that in the case of standard second-order equations modelled by a vector field Γ on some first-jet space, symmetries are invariant vector fields X, i.e. we have $\mathcal{L}_\Gamma X = 0$, whereas adjoint symmetries can be thought of (see [10]) as 1-forms α with the properties: $\mathcal{L}_\Gamma \alpha = 0$, $\langle \Gamma, \alpha\rangle = 0$. Roughly speaking, symmetries are related to solutions of the linear variational equations and adjoint symmetries to solutions of their adjoints. These solutions in fact define half of the components of the corresponding geometrical object (the other half are then automatically determined). The geometrical interpretation of this feature is precisely that both the horizontal and vertical part of a symmetry come from a single vector field along the projection on the 'configuration space' (likewise for adjoint symmetries).

Returning to our present situation, let $Z \in \mathcal{X}(J^1_\sigma)$ be a symmetry of Γ: $[Z, \Gamma] = 0$. Z can locally be written as

$$Z = \mu^\alpha X_\alpha + \mu^a \frac{\partial}{\partial q^a} + \nu^\alpha \frac{\partial}{\partial \dot{q}^\alpha}.$$

The symmetry requirement imposes conditions on the components (μ^α, μ^a) and further uniquely determines the ν^α in terms of the μ^α. Essentially therefore, we are looking at conditions on a vector field $X = \overline{X} + \widetilde{X} \in \mathcal{X}(\rho)$ and Z in fact can be decomposed in the form

$$Z = \overline{X}^H + \widetilde{X}^H + \overline{Y}^V,$$

where $\overline{Y} \in \overline{\mathcal{X}}(\rho)$ is known once we have found X. To be precise, one easily verifies in coordinates that $[Z, \Gamma] = 0$ is equivalent to the two conditions

$$\nabla^2 \overline{X} + \Phi(X) = 0,$$

$$\nabla \tilde{X} + \Psi(X) = 0,$$

and these in turn can equivalently be grasped into the single requirement

$$\nabla^2 \overline{X} + \nabla \widetilde{X} + (\Phi + \Psi)(X) = 0.$$

Concerning the dual picture, observe first that a 1-form $\phi \in \bigwedge^1(J^1_\sigma)$ for which $\langle \Gamma, \phi \rangle = 0$ is of the form

$$\phi = a_\alpha \omega^\alpha + c_a \eta^a + b_\alpha \theta^\alpha,$$

where the ω^α are the 'force forms': $\omega^\alpha = d\dot{q}^\alpha - f^\alpha dt$. This time, $\mathcal{L}_\Gamma \phi = 0$ creates differential conditions on the functions (a_α, c_a) and further fixes the b_α in terms of the a_α. Again, the essential part of the problem relates to some element $\alpha = \overline{\alpha} + \widetilde{\alpha} \in \bigwedge^1(\rho)$ and ϕ has a decomposition of the form

$$\phi = \overline{\alpha}^V + \widetilde{\alpha}^H + \overline{\beta}^H,$$

for some $\overline{\beta}$ which is further not of interest. One verifies that $\mathcal{L}_\Gamma \phi = 0$ if and only if

$$\nabla^2 \overline{\alpha} - \nabla \tilde{\alpha} + (\Phi + \Psi)(\alpha) = 0.$$

This is clearly the adjoint equation of the above requirement for symmetries and we have obtained the perfect analogue of the picture for unconstrained second-order equations (see [11]) and of the situation for the case of linear constraints as reported in [8]. By analogy with these previous situations, the tensor field $\Phi + \Psi$ could be called the *Jacobi endomorphism* for the present theory.

For 'pure' second-order equations, adjoint symmetries are known to generate, under appropriate circumstances, a Lagrangian for the system or a first integral (in a way generalizing Noether's theorem). A similar result was obtained already in the case of linear constraints. Let us mention without proof what the analogue here will look like.

Let ϕ be an adjoint symmetry of Γ, determined by an element $\alpha \in \bigwedge^1(\rho)$ of the form

$$\alpha = \overline{\alpha} + \widetilde{\alpha} = a_\alpha \theta^\alpha + c_a \eta^a,$$

and assume now that

$$a_\alpha = \frac{\partial F}{\partial \dot{q}^\alpha}, \quad c_a = \frac{\partial F}{\partial q^a} \qquad \text{for some function } F.$$

Then, if the function $L^* = \Gamma(F)$ has regular Hessian (with respect to the $\dot{q}^\alpha$), L^* is a Lagrangian for the second-order part of the system, meaning that the equations $\ddot{q}^\alpha = f^\alpha(t, q^A, \dot{q}^\beta)$ do not depend on the q^a and are equivalent to the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \dot{q}^\alpha}\right) - \frac{\partial L^*}{\partial q^\alpha} = 0.$$

The situation where we encounter a sort of generalization of Noether's theorem is in some sense a special case of this result. In fact it corresponds to the case that $\Gamma(F)$ happens to be zero, so that F is a first integral. Conversely also, every first integral can be related to an adjoint symmetry in this way.

The results reported so far have largely been obtained via calculations in local coordinates. Work is underway and will be reported elsewhere concerning coordinate free constructions of the most relevant concepts and geometrical objects in this theory. It will, in particular, be of interest to have a good view and understanding of different possible paths which can lead to the identification of the important dynamical covariant derivative ∇ and its role in applications.

To close the present discussion, we will briefly sketch a different type of application of our approach to the geometry of mixed first and second-order equations in the next section.

6. Application: non-holonomic mechanics

Let us go back to the way mixed equations can arise from the mechanics of Lagrangian systems with non-holonomic constraints. What we want to do here is to show how the geometrical approach can be used to design a direct construction of the right dynamical system on the constraint submanifold J^1_σ. For simplicity, consider, as in the introduction, the case of linear constraints of the form $\dot{q}^a = B^a_\alpha(t, q^A)\dot{q}^\alpha + B^a(t, q^A)$, to which a Lagrangian system on the free space $J^1\tau_1$ is being subjected.

Geometrically speaking, the data then are: a function $L \in C^\infty(J^1\tau_1)$ and a connection $\tilde{\sigma}$ on π. As a preliminary remark, observe that the tensor field

$$\Psi = C^a_\alpha\, \theta^\alpha \otimes \frac{\partial}{\partial q^a}$$

on J^1_σ can be 'lifted' to a tensor field on $J^1\tau_1$ with coordinate expression

$$\dot{\Psi} = C^a_\alpha\, \theta^\alpha \otimes \frac{\partial}{\partial \dot{q}^a}.$$

A direct construction of the dynamical vector field Γ on J^1_σ then proceeds as follows:

- Put $\overline{L} = i^* L$, where $i : J^1_\sigma \to J^1\tau_1$ is the injection map.

- Define two 1-forms on J^1_σ:

$$\begin{aligned} \theta_{\overline{L}} &= \overline{L}dt + S(d\overline{L}), \\ \psi_{(L,\sigma)} &= i^*\Big(\dot{\Psi}(dL)\Big) - N(d\overline{L}). \end{aligned}$$

- Define the fundamental 2-form of non-holonomic Lagrangian mechanics to be:

$$\Omega = d\theta_{\overline{L}} + \psi_{(L,\sigma)} \wedge dt.$$

- Finally, define the dynamics to be governed by the unique SODE Γ on J^1_σ, determined by the condition $i_\Gamma \Omega = 0$.

The justification for this procedure is simply that this Γ produces the correct equations of motion (see Section 1). Moreover, as was discussed in [9], the procedure still works in the case of Lagrangian systems with general, non-linear constraints.

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THEORETICAL MECHANICS DIVISION, UNIVERSITY OF GHENT
KRIJGSLAAN 281-S9, B-9000 GHENT, BELGIUM