DIFFERENTIAL GEOMETRY AND APPLICATIONS
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DIFFERENTIAL GEOMETRY AND ELEMENTARY PARTICLES

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1. Introduction

Applications of the differential geometry to theoretical physics may be broadly divided into two classes.

In the first class the differential geometry plays a role of a mathematical tool that helps to formulate and solve problems in a basically finished physical theory. Classical newtonian mechanics is the best example. The theory itself is complete, but it contains number of unsolved problems which become more transparent when the geometrical concepts like symplectic manifolds and associated structures are introduced.

The second class of applications is more concrete in the sense that the geometrical structures are closely related to the structure of the physical world. The main example is Einstein's theory of gravitation. We do not live in symplectic manifolds of the classical mechanics, they just happen to fit the mathematical structure of the theory, but in certain sense we do live in the space-time of the theory of relativity. It is this type of applications of the differential geometry which I want to address in the present contribution.

The remarkable success of Einstein's theory and its ties with the differential geometry generated a strong feeling among scientists that Physics as such is in fact a branch of applied differential geometry. It has been known since the beginning of the twentieth century that the world is fundamentally composed from fields and particles. For the sake of unified description many physicists tried to think of particles as of special or singular field configuration. Einstein himself tried to represent a particle by Schwarzschild solution of his equations. It was already known that particles behave according to the strange rules of quantum theory. Einstein hoped that the quantum behaviour could somehow be derived from the gravitational field equations which, due to their nonlinear character, could in principle lead to unexpected results.[1] His attempts remained unsuccessful, and today it is clear why. I want to list two main reasons.

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Firstly, a singularity of a classical field like in the Schwarzschild solution will always have a definite position in space, which precludes explanation of the double-slit experiment performed with individual particles. In Einstein's time such experiments were always performed with particle beams where the interference was understood to be of one particle with another, while in the present experiments the single particle must somehow pass through both slits in order to interfere with itself.

Secondly, it is difficult to see how the spin of fermions could be faithfully represented using only the real four-dimensional space-time geometry (by what I do not want to say that people never tried it). In order to describe fermions under spatial rotations one has to introduce complex vectors which do not form a natural part of the original geometry.

If one maintains that the whole physical world including both fields and particles is to be described by geometrical properties of some fundamental continuum, then one has to search for such a continuum beyond the space-time manifold. In fact, one should expect that the existence of the space-time manifold itself is only a manifestation of the properties of the fundamental continuum, a manifestation that depends on the method of observation. In the present contribution I want to sketch a few main points of such a search, concentrating on the role of the differential geometry.

2. Linear connections in real manifolds

In this section I shall introduce my notation and the method of approach regarding linear connections in real manifolds which will be generalized to connections in complex manifolds later on.

The fundamental geometrical concept is an n-dimensional real manifold M and the bundle of linear frames $P(M, GL(n, \mathbf{R}))$[2]. One defines a local reference cross section $u : U(x) \rightarrow P$ by

$$u(x) = \{h_i^\mu(x)\frac{\partial}{\partial x^\mu}; i = 1, .., n\}$$

where the invertible matrices with elements $h_i^\mu(x)$ are smoothly defined on a neighbourhood $U(x)$ of $x \in M$.

The general linear group $GL(n, \mathbf{R})$ acts naturally on P by

$$u(x)g = \{h_i^\mu(x)a_j^i\frac{\partial}{\partial x^\mu}; j = 1, .., n\}$$

which defines a local coordinate system (x^μ, a_j^i) in the bundle manifold.[1] In terms of this coordinate system a linear connection on M is defined by the horizontal lift of tangent vectors at $x \in M$ to an arbitrary point on the fibre above x characterized by coordinates (x^μ, a_j^i):

$$X_\mu = \frac{\partial}{\partial x^\mu} - A_{\mu j}^i(x)V_i^j \tag{1}$$

[1] The usual summation convention applies

where

$$V_i^j = a_k^j \frac{\partial}{\partial a_k^i}$$

are the right invariant vector fields in $GL(n, \mathbf{R})$.

When the local coordinate system in P is changed by a gauge transformation

$$h_i^\mu \to \tilde{h}_i^\mu = h_j^\mu a_i^j(x)$$

where $a_i^j(x)$ correspond to a smooth function $U(x) \to GL(n, \mathbf{R})$, the components of the connection transform according to

$$\tilde{A}_{\mu j}^i = b_k^i(x) A_{\mu l}^k(x) a_j^l(x) - (\partial_\mu b_k^i(x)) a_j^k(x) \tag{2}$$

where the matrices with elements $b_j^i(x)$ and $a_j^i(x)$ are inverse to each other.

A particular case of the gauge transformation (2) when the resulting reference cross section is the coordinate one reads as

$$\Gamma_{\mu\nu}^\sigma(x) = h_i^\sigma(x) A_{\mu j}^i(x) h_\nu^j(x) - (\partial_\mu h_i^\sigma(x)) h_\nu^i(x) \tag{3}$$

We say that the connection is reducible to $SO(r, s)$ subgroup of $GL(n, \mathbf{R})$, $n = r + s$, when it is possible to write the horizontal lift in the form

$$X_\mu = \frac{\partial}{\partial x^\mu} - \frac{1}{2} A_\mu^{ij}(x) V_{ji} \tag{4}$$

where

$$V_{ji} = g_{jk} V_i^k - g_{ik} V_j^k$$

and $g_{ij} = diag(1, 1, .., 1, -1, -1, .., -1)$ with r positive and s negative entries. We have

$$A_\mu^{ij} = A_{\mu k}^i g^{kj} = -A_\mu^{ji} = -A_{\mu k}^j g^{ki} \quad .$$

3. Five-dimensional real base manifold

Let us consider a five-dimensional real manifold with a linear connection reducible to one of the de Sitter groups $SO(4,1)$ or $SO(3,2)$. We shall write the horizontal lift (4) as

$$X_\mu = \frac{\partial}{\partial x^\mu} - \frac{1}{2} A_\mu^{ij}(x) V_{ji} - A_\mu^{i5}(x) V_{5i} \tag{5}$$

where $\mu = 1, .., 5$, but $i, j = 1, .., 4$. Under gauge transformations limited to the Lorentz subgroup of the de Sitter group the connection components transform according to

$$\tilde{A}_\mu^{ij}(x) = b_k^i(x) A_\mu^{kl}(x) b_l^j(x) - (\partial_\mu b_k^i(x)) b_l^j g^{kl} \tag{6}$$

and

$$\tilde{A}_\mu^{i5}(x) = b_k^i(x) A_\mu^{k5}(x) \quad . \tag{7}$$

The gauge transformation (7) of $A^{k5}_\mu(x)$ is identical to the transformation of the inverse tetrad h^k_μ. This simple observation leads us to a possible way in which the properties of the spacetime might be derived from the properties of the fundamental continuum.

We assume the following : The fundamental continuum is the base manifold of its bundle of linear frames, but it is not open to direct observations. The only way we can investigate the geometrical properties of the fundamental continuum (including its dimension) is to set up a reference cross section in its bundle of frames, and then use the action of the structure group to measure the deviation of our selected cross section from the horizontal direction. In addition, we assume that our choice of the reference cross section is limited by the classical method of observations we have to use. Namely, we assume that our choice is free as far as the Lorentz gauge is concerned, but outside the Lorentz subgroup the gauge is fixed, so that we are not even aware of the additional gauge freedom. We can see that it is not such a strange thought if we look at the Lorentz group itself. Within the subgroup of spatial rotations the choice of the gauge is certainly entirely free, but as far as the Lorentz boosts are concerned we are limited by technical reasons to a narrow range of the parameter v/c. Before the invention of the particle accelerators any practical setup of an observational reference cross section would be almost entirely limited to the subgroup of spatial rotations.

The deviation of the reference cross section from the horizontal direction is measured by the connection components A^{ij}_μ and A^{i5}_μ. The deviations measured by A^{i5}_μ may be interpreted by the observer as translations. This is mathematically expressed by writing

$$A^{k5}_\mu(x) = \frac{1}{l} h^k_\mu(x) \tag{8}$$

where l is a universal unit of length which translates the angle of rotation into a measured distance. As long as we are unable to perform transformations outside the Lorentz group we are also unable to decide that A^{k5}_μ are in fact connection components corresponding to rotations (or pseudorotations) and not the tetrads defining the reference cross section on a four-dimensional manifold.

In this interpretation the absence of A^{k5}_μ in any given direction implies that the direction is closed as far as the translations are concerned. In this way a reduction of dimension occurs.

Let me give a particular example. Consider a connection with

$$A^{ij}_5 = 0 \ , \ A^{i5}_5 = 0 \tag{9}$$

together with (8) for $\mu = 1,..,4$. The trivial horizontal lift of the $\partial/\partial x^5$ direction reduces the observable dimension to four, while $A^{ij}_\mu(x)$ and $A^{i5}_\mu(x)$ define the connection components and the reference cross section on the four-dimensional submanifold defined by $x^5 = const$. We shall now guess geometry of the original five-dimensional manifold which can yield a reasonable four-dimensional spacetime. From the non-commutativity of de Sitter rotations it is clear that even the flat Minkowski spacetime shall correspond to a non-flat original connection. So we should try equations that do not admit flat solutions, e.g.

$$R_{\mu\nu} = \frac{4}{l^2} g_{\mu\nu} \tag{10}$$

We shall select the pentads in such a way that the 4×4 part obeys (8), while $h^5_\mu = 0$, $\mu = 1,..,4$, $h^i_5 = 0$, $i = 1,..,4$, and $h^5_5 = 1$. Assuming zero torsion

$$\Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu} \tag{11}$$

where the indices range from 1 to 5, and using formula (3) one obtains

$$h^i_\mu(x^\nu, x^5) = h^i_\mu(x^\nu, 0) e^{\pm x^5/l}$$

where the + and - signs correspond to $SO(4,1)$ and $SO(3,2)$ cases. Calculating the Ricci tensor yields

$$\begin{aligned} R_{\mu\nu} &= R^{(0)}_{\mu\nu} + \frac{4}{l^2} g_{\mu\nu} \\ R_{\mu 5} &= R_{5\mu} = 0 \\ R_{55} &= \frac{4}{l^2} g_{55} \end{aligned}$$

where $R^{(0)}_{\mu\nu}$ denotes the Ricci tensor induced on the four-dimensional submanifold. Thus the equation (10) combined with the torsion free condition and a special choice of the original reference cross section leads to spacetimes satisfying the vacuum Einstein equation.

The fact that even a flat connection is induced by a non-flat original connection is quite significant. It implies that even with the use of full range of de Sitter transformations it is impossible to make all A^{i5}_μ, $\mu = 1,..,5$ equal to zero, and thus eliminate all observable dimensions. In fact, it was shown that in general all dimensions but one may be eliminated by de Sitter gauge transformations, and that in the $SO(3,2)$ case the remaining dimension must be time-like. This has been shown for an abstractly defined de Sitter structured principle fibre bundle, [3] but it applies also in the concrete example treated above.

It was suggested that quantum particles might be using exactly this extreme gauge and that is why they do not distinguish between distinct points in space. At the same time, the progress along the remaining direction is marked by a rotation within the (4,5) plane, not unlike the rotation within the complex plane of quantum mechanics which accompanies even particles at rest. More detailed discussion of the relationship between the above described geometrical model and quantum theory may be found in ref. [4].

Let me call the attention also to the fact that in addition to the possibility of generating four-dimensional spacetimes satisfying the vacuum Einstein equation, Eq. (10) has also natural "particle-like" solutions, namely five-dimensional pseudo-spheres of radius l. It is not clear, however, how the two types of solutions can be smoothly joined together.

Generating spacetime from connections in real base manifolds still does not answer the question of the existence of spin $\frac{1}{2}$ particles. Even though the complex plane of quantum mechanics might be identified as the plane of (4,5) rotations, spin $\frac{1}{2}$ particles need more than that. They need complex vectors transforming as spinors. The easiest way to ensure that such vectors form a natural part of the geometry is to start with a complex fundamental continuum.

4. Complex base manifolds

Consider an n-dimensional complex manifold with local coordinates z^μ , $\mu = 1, .., n$, and a connection defined by the horizontal lift

$$Z_\mu = \frac{\partial}{\partial z^\mu} - A^i_{\mu j}(z) V^j_i - B^i_{\mu j}(z) \bar{V}^j_i \tag{12}$$

where

$$V^j_i = a^j_k \frac{\partial}{\partial a^i_k} \quad , \quad \bar{V}^j_i = \bar{a}^j_k \frac{\partial}{\partial \bar{a}^i_k}$$

are the right invariant vector fields of $GL(n, \mathbf{C})$. Formula (12) is the complex analog of (1). Note that the dimension of $GL(n, \mathbf{C})$ is $2n^2$.

This time we consider reductions to subgroups of $U(r, s)$ type, when the horizontal lift may be written as

$$Z_\mu = \frac{\partial}{\partial z^\mu} - A^{ij}_\mu(z) V_{ji}$$

where

$$V_{ji} = g_{jk} V^k_i - g_{ik} \bar{V}^k_j$$

and g_{ij} is the same as in the real case. This time, however, A^{ij}_μ are not skew-symmetric and we have

$$A^{ij}_\mu = A^i_{\mu k} g^{kj} = -B^j_{\mu k} g^{ki} \quad .$$

An n-dimensional complex manifold has $2n$ real dimensions. However, an additional requirement imposed on the connection components of the reduced connection leads to a natural reduction from $2n$ to n real dimensions. Namely, when

$$A^i_{\mu j} = \bar{B}^i_{\mu j}$$

which may be written also as

$$A^{ij}_\mu = -\bar{A}^{ji}_\mu$$

we have

$$\bar{Z}_\mu - Z_\mu = \frac{\partial}{\partial \bar{z}^\mu} - \frac{\partial}{\partial z^\mu} = i \frac{\partial}{\partial y^\mu}$$

where $z^\mu = x^\mu + i y^\mu$. Thus the horizontal lift is trivial in the imaginary directions and the observable manifold is reduced to the real submanifold.

In order to have a set of the right invariant vector fields that can correspond to rotations and translations on the real submanifold as in the previous section, the structure group $U(r,s)$ must contain a subgroup H locally isomorphic to a real group $SO(k,l)$ with $k+l=r+s+1$. The extra dimension is to provide the additional rotations or pseudorotations that play the role of translations in the partially fixed gauge formulation. The above condition may be satisfied only in two cases :

(1) $n=2$, where $U(2)$ contains $SU(2)$, locally isomorphic to $SO(3)$,

(2) $n=4$, where $U(2,2)$ contains two such subgroups, locally isomorphic to $SO(3,2)$ and $SO(4,1)$.

Thus when God was deciding how to construct the world and wanted to follow such a scheme, He or She had only two choices : Either have the observable space as two-dimensional without any time, or four-dimensional with a possibility of having a time coordinate.

Tangent vectors of the base manifold transform according to the action of the complex group and in the four-dimensional case under the action of the subgroup locally isomorphic to $SO(3,1)$ they transform like Dirac spinors.

5. Discussion and perspectives

If the real four-dimensional spacetime were indeed a result of specific classical observations made on a more general continuum, the approach to a number of contemporary problems would have to be modified. For example, the canonical quantization of gravitation may be simply non-existent. Gravitation is an expression of the spacetime geometry and if the quantum effects are connected with leaving the classical gauge, the spacetime itself and with it the theory of gravitation may lose their meaning after such move. This may explain why after so many years of hard work by a number of excellent researchers the quantum theory of gravity still does not exist.

In order to plan further research work one should speculate on how the complex vectors may enter the observed real spacetime. Local gauge transformations act via the adjoint representation and the particular transformation properties of the complex tangent vectors are thus lost in the process of local construction of the real spacetime. However, depending on the global structure of the base manifold, it may be necessary to join together reference cross sections over different regions and that is where the original complex group may demonstrate its presence. In other words, the existence of spin $\frac{1}{2}$ particles may be connected with the global structure of the principal fibre bundle consisting of linear frames of the complex base manifold.

The next step should be the construction of an equation for the metric of the complex manifold that would play the role of Eq. (10) proposed for the metric of the five-dimensional real manifold. One must expect that the solutions of such an equation allow the possibility of generating spacetimes satisfying Einstein's equation, including the flat Minkowski spacetime.

A related question is how to generalize the torsion free condition from the real case to the complex one. A preliminary investigation indicates that a condition that would make the base manifold into a Kaehler manifold is too restrictive and does not allow generation of the flat space.

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