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THE GEOMETRY OF LIGHTLIKE HYPERSURFACES ON MANIFOLDS ENDOWED WITH A CONFORMAL STRUCTURE OF LORENTZIAN SIGNATURE

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ABSTRACT. The authors study the geometry of lightlike hypersurfaces V on a manifold (M,c) endowed with a pseudoconformal structure c of Lorentzian signature. They find singular points and singular submanifolds of V and use them to construct a conformally invariant normalization of V intrinsically connected with its geometry. This normalization is constructed by means of the elements of a fourth-order neighborhood of a point $x \in V$ and induces a torsion-free affine connection on V which is also intrinsically connected with the geometry of V.

1. The pseudo-Riemannian manifolds (M,g), $\dim M = 4$, sign $g = (3,1)$, is a model of spacetime of general relativity. For $\dim M = n > 4$ and sign $g = (n-1,1)$, we obtain natural generalizations of this model. Lightlike hypersurfaces in (M,g) are models of different types of horizons separating domains of (M,g) with different physical properties.

Many events and objects of general relativity are invariant under conformal transformations of the metric g. Hence it is appropriate to study them on a manifold (M,g), $\dim M = n$, endowed with a conformal structure $c = CO(n-1,1)$ of Lorentzian signature, $c = \{\sigma g\}$, $\sigma = \sigma(x) \neq 0$, $x \in M$.

A hypersurface on (M,g) is called lightlike if it is tangent to the light cone C_x at each of its points x. Since the light cones on (M,g) are conformally invariant, then the same is true for a lightlike hypersurface. Thus it is appropriate to study lightlike hypersurfaces on the manifold (M,c).

In the present paper we study singular points and singular submanifolds of a lightlike hypersurface $V \subset (M,c)$ and use them to construct a conformally invariant normalization of V intrinsically connected with V. This normalization induces a torsion-free affine connection on V which is also intrinsically connected with the geometry of V.

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The results and methods of this paper can be used for studying lightlike hypersurfaces in the pseudo-Riemannian space (M, g) of Lorentzian signature, the pseudoconformal space C_1^n, and the Minkowski space R_1^n.

Lightlike hypersurface V on a manifold (M, g) of Lorentzian signature and in the space R_1^n were considered by many authors (see the books [DB 96] and [K 96] where one can find further references). However, an invariant normalization intrinsically connected with the geometry of V was not constructed in all these papers.

As we show in the present paper (see also our papers [AG 98a] and [AG 98b]), such normalizations can be constructed only by means of elements of a neighborhood of a point $x \in V$ of order not less than three. But in the papers mentioned above a normalization was constructed by means of some normalizing elements not intrinsically connected with the geometry of V, and also by means of some elements defined in neighborhoods of the first and second order.

The more detailed proofs of the results of this paper will appear in [AG t. a.].

2. A conformal structure (M, c) of Lorentzian signature on a manifold M is given by a relatively invariant differential quadratic form g, sign $g = (n-1, 1)$. The tangent space $T_x(M, c)$ to this manifold can be compactified in such a way that it will be endowed with the structure of a pseudoconformal space $(C_1^n)_x$ of the same signature $(n-1, 1)$ (see [AG 96], Ch. 1).

Under the Darboux mapping, to the conformal space $(C_1^n)_x$ there corresponds a hyperquadric Q_1^n of a projective space P_x^{n+1}, $Q_1^n \subset P_x^{n+1}$. The equation of this hyperquadric is $(x, x) = 0$.

In the space P_x^{n+1}, we consider a family of projective frames $\mathcal{R}_x = \{A_0, A_i, A_{n+1}\}$, $i = 1, \ldots, n$, consisting of the points $A_0 = x$ and $A_{n+1} = y$ lying on the hyperquadric Q_1^n, so that

$$(A_0, A_0) = 0, \ \ (A_{n+1}, A_{n+1}) = 0\,, \ \ (A_0, A_{n+1}) \neq 0\,, \tag{1}$$

and points $A_i \in T_x \cap T_y$, where T_x and T_y are the subspaces tangent to Q_1^n at the points x and y. Besides relations (1), elements of this frame satisfy the relations

$$(A_0, A_i) = (A_{n+1}, A_i) = 0, \ \ (A_i, A_j) = g_{ij}\,, \tag{2}$$

where $\det(g_{ij}) \neq 0$, sign $g = (n-1, 1)$. The equation of the hyperquadric Q_1^n with respect to this frame has the form

$$g_{ij} x^i x^j - 2x^0 x^n = 0\,.$$

The isotropic cone C_x with vertex x is the intersection $C_x = Q_1^n \cap T_x$ and is defined by the equation $g = g_{ij}\xi^i\xi^j = 0$, $\xi \in T_x$. The group $G = \mathbf{SO}(n-1, 1) \times \mathbf{H}$, where $\mathbf{H} = \mathbf{R}^+$, leaves the cone $C_x \subset T_x$ invariant.

The equations of infinitesimal displacement of the frame $\mathcal{R}_x$ on (M, c) have the form

$$
(3) \qquad \begin{cases} dA_0 = \omega_0^0 A_0 + \omega_0^i A_i \,, \\ dA_i = \omega_i^0 A_0 + \omega_i^j A_j + \omega_i^{n+1} A_{n+1} \,, \\ dA_{n+1} = \qquad \omega_{n+1}^i A_i + \omega_{n+1}^{n+1} A_{n+1} \,. \end{cases}
$$

The 1-forms $\omega_0^i = \omega^i$ are basis forms of the manifold M. By conditions (1) and (2), the forms occurring in (3) satisfy the equations

$$
(4) \qquad \begin{cases} \omega_i^{n+1} = g_{ij}\omega_0^j \,, \;\; \omega_{n+1}^i = g^{ij}\omega_j^0 \,, \;\; \omega_{n+1}^{n+1} = -\omega_0^0 \,, \\ dg_{ij} - g_{ik}\omega_j^k - g_{kj}\omega_i^k = 0 \,, \end{cases}
$$

where $g^{ik}g_{kj} = \delta_j^i$. Note that the quantities g_{ij} and the forms ω^i are defined in a first-order neighborhood of a point $x \in V$, the forms ω_0^0 and ω_j^i in its second-order neighborhood, and the forms ω_j^0 in its third-order neighborhood.

3. The independent 1-forms occurring in equations (3) satisfy the following structure equations (see [AG 96], Section **4.1**):

$$
(5) \qquad \begin{cases} d\omega^i = \omega_0^0 \wedge \omega^i + \omega^j \wedge \omega_j^i \,, \\ d\omega_0^0 = \omega^i \wedge \omega_i^0, \\ d\omega_j^i = \omega_j^0 \wedge \omega^i + \omega_j^k \wedge \omega_k^i + g_{jk}\omega^k \wedge g^{il}\omega_l^0 + C^i_{jkl}\omega^k \wedge \omega^l \,, \\ d\omega_i^0 = \omega_i^0 \wedge \omega_0^0 + \omega_i^j \wedge \omega_j^0 + C_{ijk}\omega^j \wedge \omega^k \,. \end{cases}
$$

Here the quantities C^i_{jkl} form the *tensor of conformal curvature* of the manifold (M, c) which is also called the *Weyl tensor*. It is defined in a third-order neighborhood of a point $x \in V$. The quantities C_{ijk} are defined in its fourth-order neighborhood and do not form a tensor. Let $C = (C^i_{jkl}, C_{ijk})$. For $C = 0$, a manifold (M, c) is conformally flat. Its structure is equivalent to that of the pseudoconformal space C_1^n.

For $n \geq 4$, the condition $C^i_{jkl} = 0$ implies $C_{ijk} = 0$, and a manifold (M, c) is equivalent to the space C_1^n. For $n = 3$, the Weyl tensor $C^i_{jkl} = 0$, and the quantities C_{ijk} constitute the tensor of conformal curvature. We will not consider this case in the current paper.

The components of the Weyl tensor and the quantities C_{ijk} satisfy the conditions

$$
(6) \qquad \begin{cases} C_{ijkl} = g_{im}C^m_{jkl} \,, \\ C_{ijkl} = -C_{jikl} = -C_{ijlk}, \;\; C_{ijkl} = C_{klij} \,, \\ C_{ijkl} + C_{iklj} + C_{iljk} = 0, \;\; C^i_{jki} = 0 \,, \\ C_{ijk} = -C_{ikj}. \end{cases}
$$

For $C = (C^i_{jkl}, C_{ijk}) = 0$, equations (3) are completely integrable and define a conformal frame bundle in the space C_1^n. For $C \neq 0$, they are not completely integrable, but they can be integrated along any smooth curve $x = x(t)$. As a result, we obtain a *development* of the manifold (M, c) into C_1^n along the curve $x = x(t)$.

Isotropic geodesics are invariant under a conformal transformation of the metric g (see [AG 96], pp. 138–140). Thus they can be considered on a manifold (M, c) endowed with a conformal structure. Under the development of (M, c) into C_1^n along an isotropic geodesic, to this isotropic geodesic there corresponds a rectilinear generator of the hyperquadric $Q_1^n \sim (C_1^n)_x$.

4. A hypersurface $V \subset (M, c)$, $\dim V = n - 1$, is called *lightlike* if it is tangent to the light cone C_x at each point $x \in V$. With each point $x \in V$ we associate a frame such that $x = A_0$, the straight line $A_0A_1 \subset T_x(V)$ is tangent to V along the isotropic direction, and $A_a \subset T_x(V)$, $a = 2, \dots, n-1$. The subspace $S_x = A_0 \wedge A_2 \wedge \dots \wedge A_{n-1} \subset T_x(V)$, $\dim S_x = n - 2$, is called the *screen subspace* of V. We will take a point $A_n \in T_x(V)$ in such a way that the straight line A_0A_n is a second isotropic straight line that is conjugate to the screen subspace S_x. This line A_0A_n is called a *normalizing straight line*. We normalize the points A_1 and A_n by the condition $(A_1, A_n) = -1$.

In this frame we have

$$(7) \qquad (g_{ij}) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & g_{ab} & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad i, j = 1, \dots, n; \ a, b = 2, \dots, n-2,$$

and the equation of the isotropic cone C_x takes the form

$$g_{ab}\xi^a\xi^b - 2\xi^1\xi^n = 0, \ \ g_{ab}\xi^a\xi^b > 0,$$

where $\xi = \{\xi^a\} \neq 0$. From (3) and (7) it follows that

$$(8) \qquad \begin{cases} \omega_1^n = \omega_n^1 = 0, & \omega_1^1 = -\omega_n^n, \\ \omega_a^1 = g_{ab}\omega_n^b, & \omega_a^n = g_{ab}\omega_1^b, \\ dg_{ab} - g_{ac}\omega_b^c - g_{cb}\omega_a^c = 0. \end{cases}$$

A frame fiber bundle associated with a hypersurface V we have constructed is a first-order frame bundle. With respect to this frame bundle the hypersurface V is defined by the Pfaffian equation

$$(9) \qquad \omega_0^n = 0.$$

Thus the first equation of (3) becomes

$$(10) \qquad dA_0 = \omega_0^0 A_0 + \omega_0^1 A_1 + \omega_0^a A_a,$$

where the forms ω^1 and ω^a are basis forms of the hypersurface V.

On the hypersurface V, the first fundamental form takes the form

$$g|_V = (dA_0, dA_0) = g_{ab}\omega^a\omega^b,$$

and sign $g|_V = (n-2, 0)$. The equations

$$\omega^a = 0$$

define a foliation of isotropic lines on V.

Differential prolongation of equation (9) leads us to the following equations

$$\omega_a^n = \lambda_{ab}\omega^b, \quad \lambda_{ab} = \lambda_{ba}\,. \tag{11}$$

The investigation of equation (9) and its differential prolongation (11) leads to the following result.

Theorem 1 *Lightlike hypersurfaces V on a manifold (M, c) endowed with a conformal structure exist, and the solution of a system defining such hypersurfaces depends on one function of $n-2$ variables.*

Note that by means of equations of (4) and formulas (8), equations (11) are equivalent to the equations

$$\omega_1^a = \lambda_b^a\omega^b, \quad \lambda_b^a = g^{ac}\lambda_{cb}\,, \tag{12}$$

and it follows from (4) that

$$\omega_1^{n+1} = 0\,. \tag{13}$$

Equations (8) and (13) imply that

$$dA_1 = \omega_1^0 A_0 + \omega_1^1 A_1 + \omega_1^a A_a\,, \tag{14}$$

Equations (10), (12), and (14) prove the following result.

Theorem 2

(a) *$V = f(M^{n-2} \times l)$ where f is a differentiable mapping and $l \sim P^1$ is an isotropic geodesic on V.*

(b) *$J = \det(\lambda_b^a - s\delta_b^a)$ is the Jacobian of the mapping f at the point $z = A_1 - sA_0 \in l \subset V$.*

(c) *The line l carries $n-2$ real singular points $F_a = A_1 - s_a A_0$ of the mapping f, where s_a are roots of the equation $J = 0$ (we count each singular point as many times as its multiplicity).*

(d) *Under the development of V into $(Q_1^n)_x \subset P^{n+1}$, to the hyperplanes $T_z(V)$, where z is a regular point of l, there corresponds the same space T_l,* $\dim T_l = n-1$, *tangent to $(Q_1^n)_x$ along l.*

5. Consider

$$\lambda = \frac{1}{n-2}\sum_a s_a = \frac{1}{n-2}\lambda_a^a\,, \tag{15}$$

where $\lambda_a^a = \mathrm{tr}\,(\lambda_a^b)$. The quantity λ satisfies the following differential equation:

$$d\lambda - \lambda(\omega_1^1 - \omega_0^0) - \omega_1^0 = -\mu\omega^1 - \mu_a\omega^a\,. \tag{16}$$

The point $H = A_1 - \lambda A_0$ is called the *harmonic pole* of the point $x = A_0$ with respect to the singular points F_a. For $n = 4$, the point H satisfies the relation $(A_0, H; F_2, F_3) = -1$.

The quantities $h_{ab} = \lambda_{ab} - \lambda g_{ab}$ are components of a trace-free tensor of weight 1 on $V : h_{ab} g^{ab} = 0$. The tensor h_{ab} is the conformally invariant *second fundamental tensor* of V. The quadratic form $h = h_{ab}\omega^a\omega^b$ is the conformally invariant *second fundamental form* of V.

The tensor $h^a_b = g^{ac} h_{cb} = \lambda^a_b - \lambda\delta^a_b$ is called the *conformal affinor of Burali-Forte* (the conformal shape operator), tr $(h^a_b) = h^a_a = 0$.

The following theorem is valid.

Theorem 3

(a) *The harmonic pole H is a regular point on a line l if and only if* $\det\,(h^a_b) \neq 0$.

(b) *If $(h^a_b) = 0$, then V is a totally umbilical hypersurface, and H is the only singular point on the isotropic geodesic l. H describes a line γ tangent to l.*

(c) *If $(h^a_b) = 0$ and the structure (M, c) is conformally flat, then H is fixed, and V coincides with the isotropic cone $C_x \subset Q^n_1 \sim C^n_1$.*

6. As we noted earlier, the forms ω^1 and ω^a, $a = 2, \ldots, n-1$, are the basis forms of the hypersurface V. By (5), their exterior differentials have the form

$$\left\{\begin{array}{l} d\omega^1 = \omega^1 \wedge (\omega^1_1 - \omega^0_0) + \omega^a \wedge \omega^1_a\,, \\ d\omega^a = \omega^1 \wedge \omega^a_1 + \omega^b \wedge (\omega^a_b - \delta^a_b\omega^0_0)\,. \end{array}\right. \tag{17}$$

Thus the matrix 1-form

$$\omega = \begin{pmatrix} \omega^1_1 - \omega^0_0 & \omega^1_a \\ \omega^a_1 & \omega^a_b - \delta^a_b\omega^0_0 \end{pmatrix}$$

defines an affine structure on V. This matrix 1-form defines a torsion-free affine connection on V if and only if

$$d\omega + \omega \wedge \omega = \Omega\,, \tag{18}$$

where Ω is the curvature 2-form of this connection which is a linear combination of exterior products of the forms $\omega^1 \wedge \omega^a$ and $\omega^a \wedge \omega^b$ (see [KN 63], Ch. III).

By differentiating the form ω, one can prove that condition (18) holds on V if and only if the forms $\omega^0_1, \omega^0_a, \omega^0_n$, and $\omega^a_n = g^{ab}\omega^1_b$ are linear combinations of the basis forms ω^1 and ω^a. This leads to the following result.

Theorem 4 *To define a torsion-free affine connection on a lightlike hypersurface $V \subset (M, c)$, it is sufficient to assign on V a screen distribution $S = \cup S_x, x \in V$, and a field L of normalizing hyperplanes $L_x \subset T_x(M)$,* $\dim L_x = n - 1$, $x \notin L_x$.

7. We will make a reduction of the first-order frame bundle by imposing a point A_1 with the harmonic pole H of the isotropic geodesic $l, A_1 = H$. Then it follows from Section **5** that

$$\lambda = 0, \;\; \lambda_{ab} = h_{ab}\,, \;\; \lambda^a_b = h^a_b, \;\; \omega^a_1 = h^a_b\omega^b\,. \tag{19}$$

The constructed reduced frame bundle is a second-order frame bundle.

To construct normalizing elements, a screen subspace S_x and a normalizing hyperplane L_x, we consider the subspace $R_x \subset T_x(V)$, $\dim R_x = n-3$, which is complementary to the straight line $A_0A_1 \subset (P^{n+1})_x$. The subspace R_x allows us to find a screen subspace $S_x = x \wedge R_x$ and a *complementary screen subspace* $\widetilde{S}_x = H(x) \wedge R_x$.

In R_x take the basis points

$$C_a = A_a + y_a A_0 + z_a A_1\,, \quad a = 2, \dots, n-1\,. \tag{20}$$

The conditions for the subspace R_x to be invariant is

$$\nabla_\delta y_a + \pi_a^0 = 0\,, \ \ \nabla_\delta z_a + z_a(\pi_1^1 - \pi_0^0) + \pi_a^1 = 0\,. \tag{21}$$

Here δ is the symbol of differentiation with respect to fiber parameters, $\pi_\eta^\xi = \omega_\eta^\xi(\delta)$, and

$$\nabla_\delta y_a = \delta y_a - y_b(\pi_a^b - \delta_a^b \pi_0^0)\,, \ \ \nabla_\delta z_a = \delta z_a - z_b(\pi_a^b - \delta_a^b \pi_0^0)\,.$$

The objects y_a and z_a are called the *normalizing objects.*

By relations (19), equations (16) take the form

$$\omega_1^0 = \mu \omega^1 + \mu_a \omega^a\,, \tag{22}$$

where $\mu = \frac{1}{n-2} h_b^a h_a^b$ is a relative invariant defined in a second-order neighborhood of a point $x \in V$, $\mu > 0$ if $(h_a^b) \neq 0$, and the quantities μ_a are defined in a third-order neighborhood of $x \in V$.

For $\omega^1 = \omega^a = 0$, the quantities μ and μ_a satisfy the equations

$$\delta \mu - 2\mu(\pi_1^1 - \pi_0^0) = 0\,, \tag{23}$$

and

$$\nabla_\delta \mu_a - \mu_a(\pi_1^1 - \pi_0^0) + h_a^b \pi_b^0 - \mu \pi_a^1 = 0\,. \tag{24}$$

It follows from (23) that the quantity μ satisfies the equation

$$d\mu - 2\mu(\omega_1^1 - \omega_0^0) = \nu \omega^1 + \nu_a \omega^a\,, \tag{25}$$

where the quantities ν and ν_a are defined in a third-order neighborhood of a point $x \in V$.

For $\omega^1 = \omega^a = 0$, prolongation of equation (25) gives

$$\left\{ \begin{array}{l} \delta \nu - 3\nu(\pi_1^1 - \pi_0^0) = 0\,, \\ \nabla_\delta \nu_a - 2\nu_a(\pi_1^1 - \pi_0^0) + 2\mu \pi_a^0 - (2\mu h_a^b + \nu \delta_a^b)\pi_b^1 = 0\,. \end{array} \right. \tag{26}$$

Equations (23) and (26) prove that the quantities μ and ν are relative invariants of weight 2 and 3, respectively.

8. We will construct now normalizing objects of a lightlike hypersurface $V \subset (M, c)$ intrinsically connected with the geometry of V. Consider the objects

$$M_a = h_a^b \mu_b + \frac{\nu}{2\mu}\mu_a - \frac{1}{2}\nu_a\,, \quad N_a = -\mu\mu_a + \frac{1}{2}h_a^b \nu_b\,. \tag{27}$$

For $\omega^1 = \omega^a = 0$, they satisfy the equations

$$\begin{cases} \nabla_\delta M_a - 2M_a(\pi_1^1 - \pi_0^0) + H_a^b \pi_b^0 = 0\,, \\ \nabla_\delta N_a - 3N_a(\pi_1^1 - \pi_0^0) - \mu H_a^b \pi_b^1 = 0\,, \end{cases} \tag{28}$$

where

$$H_a^b = h_a^c h_c^b + \frac{\nu}{2\mu} h_a^b - \mu\delta_a^b \tag{29}$$

is the tensor defined in a third-order neighborhood of $x \in V$. For $\omega^1 = \omega^a = 0$, this tensor satisfies the equations

$$\nabla_\delta H_a^b = 2H_a^b(\pi_1^1 - \pi_0^0)\,, \tag{30}$$

and hence it is a relative tensor of weight 2.

The following theorem can be proved by a direct calculation.

Theorem 5

(a) *The tensor H_a^b is trace-free: $H_a^a = 0$.*

(b) *The tensor H_a^b can be reduced to a diagonal form simultaneously with the tensors g_{ab} and h_a^b and has the eigenvalues*

$$H_a = h_a^2 + \frac{\nu}{2\mu} h_a - \mu\,,$$

where h_a are the eigenvalues of the tensor h_a^b.

(c) *If the tensor H_a^b is degenerate, then the relative invariants μ and ν are connected by an algebraic equation, and the hypersurface V is of a special type.*

(d) *For $n = 4$, the tensors H_a^b and h_a^b are proportional: $H_a^b = \frac{\nu}{2\mu} h_a^b$.*

9. Suppose now that the tensor H_a^b is nondegenerate. Denote by $\widetilde{H}_a^b$ the inverse tensor of the tensor H_a^b. Then we can construct two more objects

$$P_a = \widetilde{H}_a^b M_b\,, \quad Q_a = \frac{1}{\mu}\widetilde{H}_a^b N_b\,.$$

By (27), (28), and (30), these objects satisfy the equations

$$\nabla_\delta P_a + \pi_a^0 = 0\,, \quad \nabla_\delta Q_a + Q_a(\pi_1^1 - \pi_0^0) + \pi_a^1 = 0\,. \tag{31}$$

Comparing (31) and (21), we see that the objects P_a and Q_a are normalizing objects, and the points

$$C_a = A_a + P_a A_0 + Q_a A_1$$

define an invariant subspace R_x intrinsically connected with the geometry of the hypersurface V. Thus the following theorem is valid.

Theorem 6 *If* $\det(H^a_b) \neq 0$, *then an invariant screen distribution* S *and an invariant complementary screen distribution* $\widetilde{S}$ *that are intrinsically connected with the geometry of* $V \subset (M, c)$ *are defined by elements of a third-order neighborhood of a point* $x \in V$.

10. We will make a reduction of the second-order frame bundle by imposing points A_a with the points C_a. As a result, we obtain a third-order frame bundle in which

$$P_a = Q_a = 0 \Longrightarrow M_a = N_a = 0 \Longrightarrow \mu_a = \nu_a = 0\,.$$

This and formulas (31) imply that the forms ω^0_a and ω^1_a are expressed in terms of the basis forms ω^1 and ω^a:

$$\left\{ \begin{array}{l} \omega^0_a = \sigma_a \omega^1 + \sigma_{ab}\omega^b\,, \\ \omega^1_a = \tau_a \omega^1 + \tau_{ab}\omega^b\,. \end{array} \right. \tag{32}$$

The coefficients $\sigma_a, \sigma_{ab}, \tau_a$, and τ_{ab} are defined by elements of a fourth-order neighborhood of a point $x \in V$.

After the indicated reduction has been made, the screen subspace $S_x = A_0 \wedge A_2 \wedge \ldots \wedge A_{n-1}$ and the normalizing straight line $A_0 A_n$, which is conjugate to the subspace S_x, with respect to the cone C_x, become intrinsically connected with the geometry of V.

In order to complete the construction of an invariant normalization of V, we need to construct an invariant point C_n on the isotropic normalizing straight line $A_0 A_n$ which together with H and R_x determines a normalizing hyperplane L_x.

The set of isotropic geodesics $\widetilde{l}$ corresponding to the straight line $A_0 A_n$ on the manifold (M, c) form an isotropic congruence U. Each ray $\widetilde{l}$ of this congruence carries $n-2$ singular points (see [AG 98a] and [AG 98b]). They are defined on $\widetilde{l}$ by the eigenvalues of the tensor τ_{ab}. As a normalizing point C_n we take the harmonic pole $\widetilde{H}$ of the point $x = A_0$ with respect these singular points. Since the tensor τ_{ab} is defined in a fourth-order neighborhood of a point $x \in V$, the point $\widetilde{H}$ and consequently the hyperplane L_x are also defined in this neighborhood.

Theorem 7 *A complete invariant normalization of a lightlike hypersurface* $V \subset (M, c)$ *intrinsically connected with the geometry of* V *is defined in a fourth-order neighborhood of a point* $x \in V$.

Corollary 8 *The curvature tensor of the affine connection induced by the complete normalization indicated in Theorem 7 is defined in a fifth-order neighborhood of a point* $x \in V$.

11. The following theorem is related to the integrability of the screen distributions.

Theorem 9

(a) *The screen distributions S and $\widetilde{S}$ are integrable simultaneously.*

(b) *The screen distributions S and $\widetilde{S}$ are integrable if and only if $\tau_{ab} = \tau_{ba}$.*

(c) *If the screen distributions S and $\widetilde{S}$ are integrable, then all singular points of the isotropic congruence U are real. The congruence U is foliated into a 1-parameter family of lightlike hypersurfaces.*

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