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CONSTRUCTION OF CMC–SURFACES IN S^3 FORM A PAIR OF NON–CONFORMAL HARMONIC MAPS INTO S^2

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ABSTRACT. We report our recent result (joint work with R. Aiyama, K. Akutagawa and M. Umehara) on a global correpondence between CMC surfaces in $P^3(c^2)$ or in $S^3(c^2)$ and a pair of non-conformal harmonic maps into S^2 in a same associated family.

Consider a problem to construct a surface with prescribed Gauss map. For surfaces in $\mathbf{R}^3$, the Gauss map is a map into S^2, and satisfies some integrability condition. Actually, let M be a Riemann surface with complex parameter z, and $g : M \to S^2$ be a map satisfying

$$H(g_{z\bar{z}} - \frac{2\bar{g}g_z g_{\bar{z}}}{1+|g|^2}) = H_z g_{\bar{z}}\,, \tag{1}$$

for a function H on M, where g is identified with g followed by a stereographic projection to $\mathbf{C}$. Kenmotsu shows that this is necessary and sufficient for the existence of a surface (possibly with singularities) $x_g : M \to \mathbf{R}^3$ with the mean curvature H and the Gauss map g, if M is contractible and $H \neq 0$ on M. In fact, x_g is represented by the so-called Kenmotsu formula [K]

$$x_g = (\int \omega_1, \int \omega_2, \int \omega_3)\,, \tag{2}$$

where

$$\omega_1 = \Re\{\psi(1-g^2)dz\}, \quad \omega_2 = \Re\{i\psi(1+g^2)dz\}, \quad \omega_3 = \Re\{2\psi g dz\},$$
$$\psi = -\frac{\bar{g_{\bar{z}}}}{H(1+|g|^2)^2}\,.$$

When g is an holomorphic map, (2) corresponds to the Weierstrass formula for minimal surfaces in $\mathbf{R}^3$. When g is a non-conformal harmonic map, H reduces to a constant and we obtain a CMC (constant mean curvature) surface by (2).

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Recently, representation formulas in spin version are obtained, for instance, by Konopelchenko [Ko], who uses the Dirac operator

$$D = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix},$$

with the half potential $U(z,\bar{z})$, which is a real valued function (essentially the mean curvature function, see (5)), to describe an integrability condition for the construction of surfaces from a given pair of complex valued functions (ψ_1, ψ_2), which is essentially the Gauss map. The surface is given by

$$\begin{aligned} x^1 &= \frac{i}{2}\int ((\bar{\psi}_2^2 + \psi_1^2)dz - (\bar{\psi}_1^2 + \psi_2^2)d\underline{a}rz), \\ x^2 &= \frac{1}{2}\int ((\bar{\psi}_2^2 - \psi_1^2)dz - (\bar{\psi}_1^2 - \psi_2^2)d\bar{z}), \\ x^3 &= -\int (\psi_1\bar{\psi}_2 dz + \bar{\psi}_1\psi_2 d\bar{z}), \end{aligned} \tag{3}$$

under the integrabiblity condition simply expressed by

$$D\psi = 0. \tag{4}$$

This corresponds to various equations in integrable systems, according to various properties of the surfaces, such as constantness of the Gaussian curvature, critical points of the Willmore functional, etc. The Gauss map is given by $\psi_1/\bar{\psi}_2$, and the metric tensor $F(z,\bar{z})^2 dz d\bar{z}$ and the mean curvature function H are given respectively by

$$F(z,\bar{z}) = |\psi_1(z,\bar{z})|^2 + |\psi_2(z,\bar{z})|^2, \quad H = 2U/F. \tag{5}$$

These expressions imply that we may consider a surface given by (2) or (3) as a surface with prescribed Gauss map as well as a surface with prescribed mean curvature function, because g and H ((ψ_1,ψ_2) and H, respectively) are related by (1) or (5), if the surface exists.

Aiyama and Akutagawa consider a similar problem in $S^3(c^2)$ and $H^3(-c^2)$, where $\pm c^2$ denote the curvature, and obtain representation formulas for surfaces with prescribed mean curvature function H [AA1,2]. We restrict our argument to a surface in $S^3(c^2)$ in this article. Here, the Gauss map means the generalized Gauss map given by $\mathcal{G} : M \to \mathrm{Gr}^+(2,2) \cong S^2(1)\times S^2(1)$, hence consists of a pair of maps $g_i : M \to S^2$, $\mathcal{G} = (g_1, g_2)$. In [AA2], they clarify the mechanism how to construct a surface in $S^3(c^2)$ from a given $\nu = g_2$ (followed by a stereographic projection) satisfying

$$\nu_{z\bar{z}} - \frac{2\bar{\nu}}{1+|\nu|^2}\nu_z\nu_{\bar{z}} = \frac{1}{H+ic}H_z\nu_{\bar{z}},$$

where H is supposed to be the mean curvature function. (Compare with (1)).

Here we recall that when H is constant, g_1 and g_2 are non-conformal harmonic maps in the same associated family [HO, S]. In fact, g_2 is related to g_1 by $g_2 = g_1^\alpha$

where g_2 has the Hopf differential multiplied by $e^{2i\alpha}$ to that of g_1, and $\cos\alpha = \sqrt{H^2/(H^2+c^2)}$ [AA2, M]. In other words, from a non-conformal harmonic map $g : M \to S^2$, the partner g^α generating a generalized Gauss map of a CMC-H surface in $S^3(c^2)$, is determined uniquely by $\cos\alpha = \sqrt{H^2/(H^2+c^2)}$. Up to now, we consider a local correpondence.

In [AAMU], we obtain a global correspondence, which is a direct consequence of the representation formula in [AA2, Theorem 4.2], but can be also proved without this formula:

Theorem. *Let $g : M \to S^2$ be a non-conformal harmonic map. Suppose that there exists a real number $\alpha \in (0, \pi/2]$ such that the associated harmonic map g^α is single-valued on M. Then there exists a branched conformal CMC-H immersion x of M into $P^3(c^2)$ with the generalized Gauss map $\mathcal{G} = (g, g^\alpha)$, where $\cos\alpha = \sqrt{H^2/(H^2+c^2)}$.*

If we take a suitable covering, the argument implies a similar results for CMC surfaces in $S^3(c^2)$.

As is well known, harmonic maps into symmetric spaces exist in an S^1 family $\{g^\alpha\}$ consisting of maps from the universal cover of M. When they are non-congruent, it is interesting to know for which spectral parameter α, g^α is single valued on M. When M is compact, those α are isolated in S^1 [AAMU]. From compact CMC surfaces in $S^3(c^2)$, we obtain a pair of such parameters. An interesting open question is whether such pair is unique or not in S^1.

Finally, we remark that Bobenko's formula [B1, Theorem 14.1] for CMC surfaces in $S^3(c^2)$, which is obtained in the same context as Sym's formula, also implies this theorem. Moreover we obtain

Corollary [B2, §5], [AAMU]**.** *Let $g : M \to S^2$ be a non-conformal harmonic map. Then the branched CMC-immersion $x_g : \tilde{M} \to \mathbf{R}^3$ with the Gauss map g given by the Kenmotsu formula* [K] *is single-valued on M if and only if $\left.\dfrac{d\rho_\theta}{d\theta}\right|_{\theta=0} = 0$ where $\rho_\theta : \pi_1(M) \to \mathrm{SU}(2)$ is the monodromy representation of g^θ.*

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