REVISITING A THEORETICAL MODEL ON FRACTIONS: IMPLICATIONS FOR TEACHING AND RESEARCH

Charalambos Y. Charalambous[1] and Demetra Pitta-Pantazi[2]

[1]University of Michigan, [2]University of Cyprus

Fractions are among the most complex mathematical concepts that children encounter in their years in primary education. One of the main factors contributing to this complexity is that fractions comprise a multifaceted notion encompassing five interrelated subconstructs (part-whole, ratio, operator, quotient and measure). During the early 1980s a theoretical model linking the five interpretations of fractions to the operations of fractions and problem solving was proposed. Since then no systematic attempt has been undertaken to provide empirical validity to this model. The present paper aimed to address this need, by analysing data of 646 fifth and sixth graders' performance on fractions using structural equation modelling. To a great extent, the analysis provided support to the assumptions of the model. Based on the findings, implications for teaching fractions and further research are drawn.

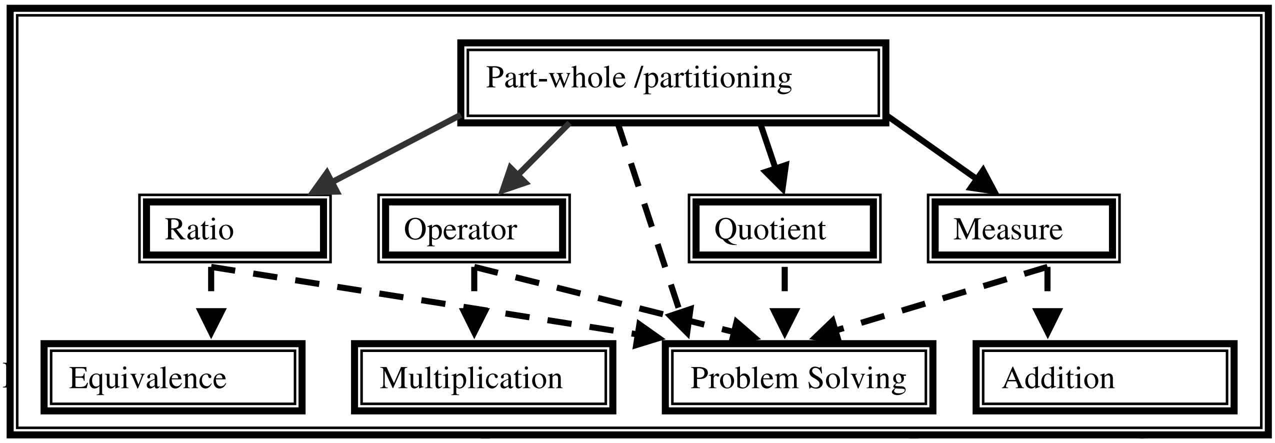
INTRODUCTION

Teaching and learning fractions has traditionally been problematic. In fact, it is well documented that fractions are among the most complex mathematical concepts that children encounter in their years in primary education (Newstead & Murray, 1998). It has also been asserted that learning fractions is probably one of the most serious obstacles to the mathematical maturation of children (Behr, Harel, Post & Lesh, 1993). During the last three decades researchers and scholars have identified several factors contributing to students' difficulties in learning fractions. In particular, it has been proposed that the obstacles that students encounter in developing deep understanding of fractions are either inherent to the nature of fractions or are due to the instructional approaches employed to teach fractions (Behr et al., 1993; Lamon, 1999). To date there is consensus among researchers that one of the predominant factors contributing to the complexities of teaching and learning fractions lies in the fact that fractions comprise a multifaceted construct (Brousseau, Brousseau & Warfield, 2004; Kieren, 1995; Lamon, 2001).

Kieren (1976) was the first to propose that the concept of fractions consists of several subconstructs and that understanding the general concept depends on gaining an understanding of each of these different meanings of fractions as well as of their confluence. Kieren initially identified four subconstructs of fractions: measure, ratio, quotient, and operator. In his original conceptualization, the notion of the part-whole relationship was considered the seedbed for the development of the other subconstructs; thereby he avoided identifying this concept as a separate, fifth, subconstruct claiming that this notion is embedded within all other subconstructs. In the following years, Behr, Lesh, Post and Silver (1983) further developed Kieren's

2005. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 233-240. Melbourne: PME.

ideas recommending that the part-whole relationship comprise a distinct subconstruct of fractions. They also connected this subconstruct with the process of partitioning. Moving a step forward, they proposed a theoretical model linking the different interpretations of fractions to the basic operations of fractions and to problem solving (Figure 1). The solid arrows presented in this proposed model suggest established relationships among fractional constructs and operations, whereas the dashed arrows depict hypothesized relationships.



Closer examination of the diagram presented in Figure 1 reveals the following. First, the part-whole subconstruct of rational numbers, along with the process of partitioning, is considered fundamental for developing understanding of the four subordinate constructs of fractions. This assumption justifies why the part-whole notion has occupied the lion's share of curricula across different countries and has been the traditional inroad to introduce fractional concepts in primary grades (Baturo, 2004; Lamon, 2001). Second, the diagram suggests that the ratio subconstruct is considered as the most "natural" to promote the concept of equivalence and, subsequently, the process of finding equivalent fractions. Moreover, the operator and measure subconstructs are regarded as helpful for developing understanding of the multiplication and addition of fractions, respectively. Finally, understanding of all five subconstructs of fractions is considered a prerequisite for solving problems in the domain of fractions.

Though the model has been excessively cited in the following years (Carpenter, Fennema & Romberg, 1993), to the best of our knowledge, no systematic attempt has been undertaken since mid 1980s to provide empirical validity to the model. The present study aimed to address this theoretical and research deficiency. Specifically, the purpose of the study was to empirically test the five theoretical assumptions of the model alluded to above and investigate the extent to which any additional associations between the concepts and operations embedded in the model are empirically supported.

THE DEVELOPMENT OF THE TEST

A test on fractions was constructed guided by existing theory and research on rational numbers. An additional requirement in designing the test was its alignment with the curriculum that was operative in Cyprus, where the study was conducted. Table 1 presents the specification table that guided the construction of the test and the items used for examining students' performance on each of the concepts and operations included in the theoretical model. Items in bold letters represent problem-solving tasks related to each of the subconstructs of fractions, since it was decided to use problems related to these subconstructs, rather than general problems on fractions.

CONCEPTS	ITEMS	OPERATIONS	ITEMS
Part-whole /partitioning	1-8, 9	*Equivalence*	34-43
Ratio	10-14, *15*	*Additive operations*	44-46
Operator	16-18, 19	*Multiplicative operations*	47-50
Quotient	20-22, 23-24		
Measure	25-31, 32-33		

Table 1: Specification table of the test used in the study

The first five items of the part-whole subconstruct asked students to identify the fractions depicted in discrete or continuous representations. The remaining three items were associated with unitizing and reunitizing, which are directly related to the partitioning notion of the part-whole relationship (Baturo, 2004). The part-whole problem-solving task (item 9) asked students to reconstruct the whole given a part of it. The subsequent five items requested students to compare ratios, based either on quantitative (10-12) or qualitative information (13-14). Item 15 referred to boys and girls sharing different numbers of pizzas; an item frequently used in studies on ratios and proportions (Marshall, 1993). The following two items asked students to specify the output quantity of an operator machine given the input quantity and the fraction operator. Item 18 was related to the notion of operator as a composite function (Behr et al., 1993), whereas item 19 asked student to indicate the factor by which number 9 should be increased to become equal to 15. In line with previous studies (Lamon, 1999), the three subsequent items, which were related to the concept of quotient, examined students' ability to link a fraction to the division of two numbers; the two problems of this category were related to the partitive and quotitive interpretation of division (items 23-24, correspondingly). In accord with previous studies (Hannula, 2003; Lamon, 1999; Marshall, 1993), the items of the measure subconstruct examined students' performance on identifying fractions as numbers and locate them on number lines. The two problems of this category asked students to find a fraction that was within two given fractions, and identify among a number of fractions the one that was closer to number one. Finally, the remaining 17 items were associated with operations on fractions. Seven of these items (41-45 and 48-49) examined students' procedural fluency in these operations, whereas the remaining ten were related to the conceptual understanding of these operations (e.g., estimating the result of different operations on fractions).

METHODS

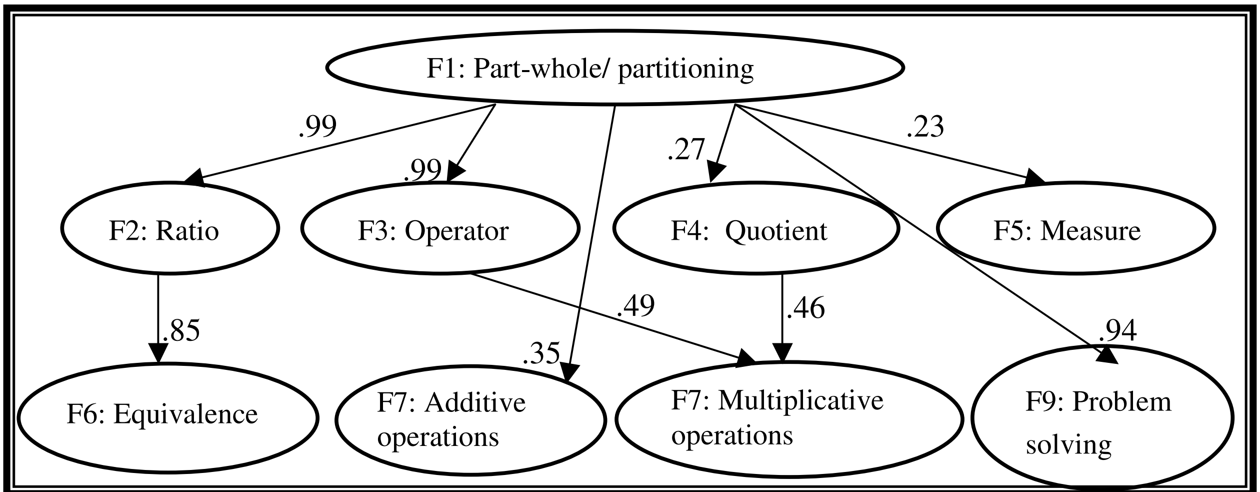
The items of the test were content validated by three experienced primary teachers and two university tutors of Mathematics Education. Based on their comments, minor amendments were made particularly where some terms used were considered as unfamiliar to primary pupils. The final version of the test (available on request) was administered to 340 5th graders and 306 6th graders (301 boys and 345 girls). The test items were split into two subtests which were administered to students during two consecutive schooldays; students had eighty minutes to work on each subtest.

Structural equation modeling and, specifically, maximum likelihood method, was used to test the hypotheses of the theoretical model (Kline, 1998). Goodness of fit of the data to the model was decided by using three fit indices: scaled x^2, Comparative Fit Index (CFI), and Root Mean Square Error of Approximation (RMSEA).

FINDINGS

The theoretical assumptions of the model were tested by using EQS. As reflected by the iterative summary, the goodness of fit statistics showed that the data did not fit the model very well (x^2=9110, df=2129, x^2/df=4.30, CFI=.57, and RMSEA=.07). Subsequent model tests revealed that the model fit indices could be improved by modifying the model in ways that on the whole were consonant with both theory on fractions and the development of the test. Specifically, items 18 and 23 were also linked to multiplicative operations, since both were solved by applying a multiplicative operation. Items 29-31 were also explained by the process of finding equivalent fractions, a relationship that could be attributed to the fact that the foregoing process provided scaffold in solving these tasks. The initial analysis also revealed that the associations of the four subordinate subconstructs of fractions with the problem solving were not significant; nor the association between the measure subconstruct and the additive operation of fractions. All these relationships were eliminated from subsequent analyses. On the contrary, the multiplicative operations of fractions were found to be associated with the quotient subconstruct, which can be explained taking into account that the preceding subconstruct is closely related to the division of fractions. The nine factor model that emerged after these modifications had a very good fit to the data (x^2=1892.55, df=1184, x^2/df=1.598, CFI=.95 and RMSEA=.030). Its goodness of fit was even better compared to a series of other models comprising of one to eight factors, thereby indicating that the emerging model was in alliance with parsimony principle (Kline, 1998). Figure 2 presents the model that emerged from the analysis; the loadings of each variable on the nine factors are presented below the model.

The following observations arise from Figure 2 and Table 1. First, all fifty items were correlated to the factors they were initially supposed to be loaded to, providing support to the construct validity of the test. However, beyond being associated with the measure notion, the three items that concerned locating fractions on number lines (items 29-31) were also related to finding equivalent fractions; their loadings in the latter case were much higher than in the former one. This finding can be partly attributed to the fact that in two of the three items the denominator of the fractions that students were requested to locate on number lines was a sub multiple of the spaces into which the given number lines were divided. Yet, one could also suggest that this finding points to the requirement that students master the equivalence of fractions in order to be able to manipulate number lines efficiently.



*EQUATIONS**: *Part-whole*: V1=.25F1, V2=.46F1, V3=.28F1, V4=.26F1, V5=.24F1, V6=.42F1, V7=.52F1, V8=.38F1 *Ratio:* V10=.38F2, V11=.44F2, V12=.25F2, V13=.34F2, V14=.29F2, *Operator*: V16=.49F3, V17=.58F3, V18=.21F3+.37F8, *Quotient*: V20=.51F4, V21=.42F4, V22=.72F4, *Measure*: V25=.46F5, V26=.98F5, V27=.98F5, V28=.93F5, V29=.11F5+.32F6, V30=.11F5+.40F6, V31=.08F5+.42F6, *Equivalence*: V34=.40F6, V35=.76F6, V36=.72F6, V37=.76F6, V38=.81F6, V39=.74F6, V40=.72F6, V41=.52F6, V42=.43F6, V43=.50F6, *Additive operations*: V44=.62F7, V45=.34F7, V46=.61F7, *Multiplicative operations*: V47=.69F8, V48=.43F8, V49=.55F8, V50=.55F8, *Problem solving*: V9=.58F9, V15=.45F9, V19=.41F9, V23=.32F9+.22F8, V24=.59F9, V32=.43F9, V33=.50F9 (*errors were omitted).

Figure 2: Path model linking the five subconstructs of fractions to the operations of fractions and to problem solving

Second, the data provided support to the assumption that the part-whole interpretation of fractions and the partitioning process constitute a foundation for developing an understanding of the four subordinate interpretations of fractions. Specifically, Factor 1 explained about 98% of the variation of the factors related to the ratio and the operator personalities of fractions (given that the percentage of the variation explained is equal to the square of the regression coefficients presented in Figure 2). Yet, one cannot ignore the fact that Factor 1 explained a much smaller percentage of the variation of the quotient and the measure notion of fractions (about 5% and 8%, respectively). Third, the data provided empirical support to the hypothesis that mastering the notion of fractions as ratios contributes predominantly to finding equivalent fractions, since Factor 2 explained a great proportion of the variation of the sixth factor (about 73%). Developing an understanding of the operator subconstruct was also found to explain about a quarter of the variation of students' performance on the items associated with the multiplicative operations on fractions.

Fourth, the model that emerged deviated from the theoretical model in three aspects: (a) the effect of the measure subconstruct on the additive operations of fractions was not statistically significant; however, a statistically significant association between the part-whole subconstruct and the additive operations emerged; (b) the associations of the four subordinate notions of fractions with problem solving were not statistically significant; on the contrary, the part-whole relationship was found to explain a great percentage of the variation of problem solving; and (c) the quotient subconstruct of fractions was found to explain about 20% of the variation of students' performance on the items related to the multiplicative operations of fractions.

In general, the model of Figure 2 verified three of the five examined hypotheses: (a) the part-whole interpretation explained a proportion of the variation of the four subordinate subconstructs of fractions; (b) the ratio notion was associated with equivalence, and (c) the operator concept was linked to the multiplicative operations of fractions. Yet, two hypotheses failed to be empirically validated. In particular, the four subordinate notions of fractions were not statistically related to problem solving, nor was the measure subconstruct related to the additive operations of fractions. Nevertheless, the study supports two additional paths not included in the theoretical model: one linking the quotient subconstruct to the multiplicative operations and the other linking the part-whole relationship to the additive operations of fractions.

DISCUSSION

The findings of the study provide empirical support to the fundamental role of the part-whole subconstruct in building understanding of the remaining constructs of fractions; thereby, they justify the traditional instructional approach in using this notion as the inroad to teaching fractions (Baturo, 2004; Kieren, 1995; Marshall, 1993). However, one cannot overlook the fact that the part-whole interpretation of fractions explains different percentages of the variation of the four subordinate concepts of fractions: almost all the variance of the ratio and the operation subconstructs and only a very small proportion of the variance of the measure and the quotient concepts. Three reasons seem to explain this finding. First, core ideas, such as comparing quantities, are embedded in all three former subconstructs of fractions, whereas they are not required for developing understanding of the latter couple of subconstructs (Lamon, 1999). Second, the measure and the quotient interpretations of fractions could be explained by other concepts not included in the model, such as the notion of the unit fraction, which is considered as contributing predominantly to building meaning for these subconstructs (Behr et al., 1983; Marshall, 1993). And third, students might encounter significant difficulties in gaining an insight into the concepts of measure and quotient, which cannot be surpassed even by developing an understanding of the part-whole "personality" of fractions. Whatever the reason is, this finding validates the claim that, the part-whole interpretation of fractions should be considered as a necessary but not sufficient condition for developing an understanding of the remaining notions of fractions (Baturo & Cooper, 1999; Brousseau et al., 2004). Thereby, though the findings of the study justify the preponderance of the part-whole interpretation of fractions in teaching rational numbers, they also underline the need for emphasizing the other subconstructs of fractions, and especially those that are not so highly related to the foregoing notion.

The study also provides support to the assumption that mastering the five interpretations of fractions contributes towards acquiring proficiency in the operations of fractions. This finding can be partly attributed to the fact that the items used for measuring students' performance on the operations of fractions required both procedural fluency and conceptual understanding of these operations; yet it also spotlights that when teaching fractions, teachers need to scaffold students to develop a profound understanding of the different interpretations of fractions, since such an understanding could also offer to uplift students' performance in tasks related to the operations of fractions. Thereby, instead of rushing to provide students with different algorithms to execute operations on fractions, the findings of the present study, in accordance with previous studies (Lamon, 1999; Brousseau et al., 2004) lend themselves to support that teachers should place more emphasis on the conceptual understanding of fractions. The study also suggests that, the teaching of the different operations of fractions should be directly linked to specific interpretations of fractions. In particular, the findings of study indicate that the teaching of equivalent fractions could be reinforced by learning ratios, whereas the operator and the quotient subconstructs could support developing a conceptual understanding of the multiplicative operations on fractions. Likewise, the associations between the part-whole interpretation and the additive operations of fractions should be highlighted during instruction, in order to promote learning of the latter processes.

Finally, one cannot ignore the fact that only a very small percentage of the variation of items related to number lines was explained by the measure subconstruct of fractions; this finding supports Lamon's (1999) recommendation that researchers employ items beyond locating fractions on number lines to measure students' understanding of fractions. In alignment with previous studies, it also suggests that the number line comprises a difficult model for students to manipulate (Baturo & Cooper, 1999), and that teachers should help students master other notions (such as the equivalence of fractions), before rushing to introduce this model in their teaching.

It goes without saying that further research is needed to cross-validate the model emerged in this study. Specifically, studies could follow at least three different directions. First, the relationships included in the theoretical model and were verified in the present study need to be further examined. Second, further studies need to verify the modifications introduced in the theoretical model, and especially the fact that only the part-whole relationship was found to be associated with problem solving. Finally, provided that the associations among the five subconstructs of fractions were verified, future research could also be directed towards identifying core ideas that permeate the whole domain of fractions and offer significantly to building understanding of all the five subconstructs included in the theoretical model.

References

Baturo, A.R. (2004). Empowering Andrea to help year-5 students construct fraction understanding. In M. J. Høines & A. Fuglestad (Eds.), *Proceedings of the 28th PME Conference, 2* (pp.95-102), Bergen: Bergen University College.

Baturo, A.R., & Cooper, T.J. (1999). Fractions, Reunitisation and the Number-Line Representation. In O. Zaslavsky (Ed.), *Proceedings of the 23rd PME Conference, 2* (pp.81-88). Haifa: Israel Institute of Technology.

Behr, M., Harel, G., Post, T. & Lesh, R. (1993). Rational Numbers: Toward a Semantic Analysis-Emphasis on the Operator Construct. In T. P. Carpenter, E. Fennema, & T.A. Romberg, (Eds.), *Rational Numbers: An Integration of Research* (pp. 13-47). NJ: Lawrence Erlbaum

Behr, M., Lesh, R., Post, T., & Silver E. (1983). Rational Number Concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of Mathematics Concepts and Processes*, (pp. 91-125). New York: Academic Press.

Bentler, P.M. (1995). *EQS-Structural Equations Program Manual.* Encino, CA: Multivariate Software, Inc.

Brousseau, G., Brousseau, N., & Warfield, V. (2004). Rationals and decimals as required in the school curriculum. Part 1: Rationals as measurements. *Journal of Mathematical Behavior, 23,* 1-20.

Carpenter, T.P., Fennema, E., & Romberg, T.A. (1993). Toward a Unified Discipline of Scientific Inquiry. In T. P. Carpenter, E. Fennema, & T. A. Romberg. (Eds.), *Rational Numbers: An Integration of Research* (pp. 1-11). NJ: Lawrence Erlbaum Associates.

Hannula, M. S. (2003). Locating Fraction on a Number Line. In N. A. Pateman, B. J. Dougherty & J. Zilliox (Eds.), *Proceedings of the 2003 Joint Meeting of the PME and PMENA, 3* (pp. 17-24). Hawaii: CRDG, College of Education, University of Hawaii.

Kieren, T.E. (1995). Creating Spaces for Learning Fractions. In J. T. Sowder and B. P. Schappelle (Eds.), *Providing a Foundation for Teaching Mathematics in the Middle Grades,* (pp. 31-66). Albany: State University of New York Press.

Kline, R.H. (1998). *Principles and Practice of Structural Equation Modeling.* London: Gilford.

Kieren, T.E. (1976). On the mathematical, cognitive, and instructional foundations of rational numbers. In R. Lesh (Ed.), *Number and Measurement: Papers from a Research Workshop* (pp. 101-144). Columbus, OH: ERIC/SMEAC.

Lamon, S.L. (2001). Presenting and Representing: From Fractions to Rational Numbers. In A. Cuoco & F. Curcio (Eds.), *The Roles Of Representations in School Mathematics-2001 Yearbook* (pp. 146-165). Reston: NCTM.

Lamon, S.J. (1999). *Teaching Fractions and Ratios for Understanding.* New Jersey: Lawrence Erlbaum Associates.

Marshall, S.P. (1993). Assessment of Rational Number Understanding: A Schema-Based Approach. In T.P. Carpenter, E. Fennema, & T.A. Romberg (Eds.), *Rational Numbers: An Integration of Research*, (pp. 261-288). New Jersey: Lawrence Erlbaum Associates.