Discrete Heisenberg-Weyl Group and Modular Group

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Consider canonical quantum mechanical operators P and Q in $L_2(\mathbf{R})$ with Heisenberg commutation relations

$$[P,Q] = -i\hbar I,$$

where I is unit operator and $\hbar$ is a real parameter (Planck constant). Discrete group, generated by I and two operators

$$u = e^{iPq/\hbar}, \quad v = e^{iQp/\hbar},$$

where p and q are fixed real numbers, depends essentially on parameter $\theta = \frac{pq}{2\pi\hbar}$ entering the Weyl relations

$$uv = e^{2\pi i\theta} vu,$$

Its group algebra will be called A_θ. The analogous algebra $A_{-1/\theta}$ is generated by I and pair of operators

$$\hat{u} = e^{\frac{2\pi iQ}{q}}, \quad \hat{v} = e^{\frac{2\pi iP}{p}}$$

with Weyl relations

$$\hat{u}\hat{v} = e^{\frac{-2\pi i}{\theta}} \hat{v}\hat{u}$$

It is evident, that algebras A_θ and $A_{-1/\theta}$ commute in $L_2(\mathbf{R})$.

Theorem. *Let θ be irrational number. Then A_θ and $A_{-1/\theta}$ are two commuting factors of type II_1, which together generate the algebra B, generated by P, Q and I,*

$$B = A_\theta \otimes A_{-1/\theta}.$$

In physical terms this means, that one degree of freedom with canonical coordinates P and Q separates into two, with coordinates u, v and $\hat{u}$, $\hat{v}$. One can call u and v compactified coordinates, related to noncompact P and Q, with θ being "radius of compactification". The theorem states that for irrational θ the dual compactified coordinates $\hat{u}$, $\hat{v}$ are to be added to u and v to generate the full algebra B.

Relation of duality $\theta \to -1/\theta$ reminds the modular transformation. In fact the action of the full modular group can be defined in B by its action on generators

$$S(u) = \hat{u}, \;\; S(v) = \hat{v}, \;\; S(\hat{u}) = u, \;\; S(\hat{v}) = v$$

$$T(u) = u, \;\; T(v) = v \;\; T(\hat{u}) = u\hat{u}^{-1}, \;\; T(\hat{v}) = v\hat{v}^{-1}$$

The relations of modular group take the form

$$S^2 = I; \;\; (ST)^3 = \sigma, \;\; \sigma(a) = a^{-1}, a = u, v, \hat{u}, \hat{v}.$$

In different setting the related results were discussed by M.Rieffel and A.Connes. Our observation on connection to the duality principle of modern quantum theory seems to be new.