The geometric compactification of the moduli of surfaces of general type (after Kollár, Shepherd-Barron, Alexeev)

Shigefumi Mori
RIMS, Kyoto University

The moduli M_g of curves of genus $g > 1$ was compactified by [Deligne, Mumford70] by adding the so called stable curves (i.e. curves with at most ordinary double points whose automorphism group is discrete).

We would like to survey the compactification of surfaces of general type due to [Kollár, Shepherd-Barron88] and [Alexeev94]. In this note, we are concerned only with the case of characteristic 0 (the field of complex numbers).

For surfaces, the compactification needed a few steps.

(1) The minimal model program (MMP).

To study the 1-parameter degeneration of n-folds, one has to consider the total space which is of dimension $n+1$. We need 3-dimensional minimal model program. The compactification needs a special but a significant case of the MMP called the semi-stable MMP (ssMMP).

The 3-dimensional MMP: The ssMMP was done by Tsunoda-Miyanishi, [Kawamata88] and Shokurov. The MMP was done by mori, and it has been generalized to log MMP [Kollár et al92], [Shokurov92].

(2) Stable surfaces and the whole plan of compactification.

[Kollár, Shepherd-Barron88] used the ssMMP to study the total space of a 1-parameter degeneration of surfaces. They defined a projective surface with only semi log canonical singularities with Q-ample K (slc surfaces) and showed that by adding these the moduli can be compactified along the 1-parameter degeneration.

They defined the functor $F_{k,\chi}$ which should have, as the coarse moduli, the compactification. The only unfinished step was to show that given k, χ the slc surfaces X with $K^2 = k, \chi(O_X) = \chi$ are bounded (i.e. parametrized by an algebric scheme).

(3) The boundedness

The boundedness was solved by [Alexeev94]. To be precise, he solved a boundedness conjecture formulated in [Kollár93].

(4) The actual construction

By the bounedness, the coarse moduli M of $F_{k,\chi}$ is expressed as the geometric quotient of certain quasi-projective scheme by a proper action of a projective linear group.

Then one can show that M is a proper algebraic space [Popp77], [Kollár95]. By [Kollár90], M is a projective scheme. ([Viehweg95] also discusses these in more details.)

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