Minimal Foliations

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In this expository lecture we want to describe some results about minimal foliations. Recently, there were a number of exciting new developments in this subject. Most interesting is the fact that they cover results in dynamical systems, in partial differential equations as well as in hyperbolic geometry. In this note, which is to be viewed as an abstract of a lecture held in Paris, May 12, 1995, we will sketch these developments and list the relevant literature. The

main theme is the observation that minimizers, or minimals, of certain variational principles play a decisive role for various problems in dynamics, in partial differential equations and for geometry. Minimals admit a certain geometric estimate which — in the terminology of Gromov [12] — is called the trapping property. These ideas enter dynamical systems in the theory developed by Aubry [2] and Mather [15, 16], in partial differential equations in the work of the author [18, 19] and in hyperbolic geometry in the recent work of Bangert and Lang [6]. For a recent survey we refer to Bangert [3] by Gromov [12]. **1.**

The geodesic flow We consider geodesics on a compact Riemannian manifold

(M^n, g), which are characterized by the property that sufficiently small arcs have minimal length. The corresponding geodesic flow φ^t takes place on the tangent bundle TM^n, and since the velocity has constant length can be restricted to the unit tangent bundle T_1M^n. This flow is called "transitive" if there exists at least one dense orbit on T_1M^n. According to an old and well-known theorem by E. Hopf and G. Hedlund the geodesic flow on a compact surface $(n = 2)$ with negative curvature is ergodic, hence transitive. This excludes the torus and the sphere and the question for smooth metrics on S^2 and T^2 with transitive flow was answered only in 1988: **Theorem 1** (V. Donnay [10]). *There*

exist smooth metrics g, such that (S^2, g), (T^2, g) resp., have transitive geodesic flows. **2. Minimal foliations on a two-dimensional torus** In the following

we will restrict ourselves to a two-dimensional torus with a Riemann- ian metric g. A family of curves covering T^2 simply and smoothly is called a "minimal foliation" if each leaf, i.e. each curve, is a geodesic. In the case of the flat torus, represented by its covering space $\mathbb{R}^2$ with the Euclidean metric, a family of parallel lines provides an example of a minimal foliation. For arbitrary metrics g such minimal foliations need not exist! This question is related to the work of Aubry [2] and Mather [15] which we illustrate with the example of the geodesic flow on T^2 (see Bangert [5]). Everything hinges on the following observation: The geodesics of a minimal foliation

1. have no selfintersection;
2. are "minimal geodesics".

A geodesic is called minimal if any of its subarcs (not only short ones) have minimal length in its homotopy class of all arcs with the same endpoints. In particular, minimal geodesics have no conjugate points. The fact i) follows from the definition and ii) follows from a well-known theorem of calculus of variations on extremal fields. In general, there may be only very few geodesics which are minimal. On the two-torus one has rather precise properties. We want to single out the "trapping property": We lift (T^2, g) to the plane $\mathbb{R}^2$ with a $\mathbb{Z}^2$-periodic metric. Then there exists a constant D, depending on the metric only, such that a given minimal geodesic is confined to a strip of width D. Conversely, any such strip contains a minimal. Consequently one can assign a well-defined direction (angle α with the x-axis) to any minimal geodesics. Hence, any minimal foliation also has a direction. Perturbation result: According to a theorem of Kolmogorov one can guarantee the existence of minimal foliations of a direction α for which α/π is a Diophantine irrational number, provided that the metric is C^∞-close to the flat metric. Hence, there is an uncountable set of minimal foliations for almost flat metrics. From this it follows that the geodesic flow is certainly not transitive: Indeed, since the lift of a minimal foliation to the unit tangent bundle of T^2 is an invariant torus of φ^t the existence of two tori confines orbits between them. If the flat metric g_0 is connected, to the metric g of theorem 1, say by $g_\lambda = (1-\lambda)g_0 + \lambda g$, $\lambda \in [0,1]$, then for some value λ all but one minimal foliation must have disappeared. This "break-down of stability" is in large part described in the work of Aubry–Mather. **3. Aubry–Mather theory** Basically

one tries to construct a minimal foliation from a given minimal by taking its closure on the torus. This can be done if the minimal is dense on the torus. This requires that α/π is irrational. In this case the lift to $T_1 M$ turns out to be a Lipschitz continuous invariant torus. If, however, the minimal is not dense and α/π irrational then its closure gives rise to a "lamination" which covers the torus only partially. The lift of this lamination defines an invariant set on the unit tangent bundle of T^2, the so-called Mather set — which locally is the product of a Cantor set and an interval. (We simplify somewhat: Actually one has to restrict oneself to recurrent minimals.) It is clear that this second case, of non-dense minimals, does occur; one can easily construct analytic metrics with this property [5]. It is interesting that this answers an old question raised by G.D. Birkhoff in 1912 (see [7]). He asked for "recurrent orbits of discontinuous type" which in modern terminology correspond to minimal sets which are locally disconnected. The existence of such minimal sets are given by Mather sets. **4. Rigidity theorems** Returning to the minimal geodesics on T^2 we note that for the flat metric all geodesics are minimal, since their lifts to $\mathbb{R}^2$ are straight lines. In 1948 E. Hopf [13] proved the remarkable result that the converse holds: If every geodesic on (T^2, g) is minimal then the metric is flat. It suffices to assume that all geodesics are free from conjugate points. For more than 40 years it has been a tantalizing problem for differential geometers to prove this statement for higher dimensions. For the history and a partial result see [9]. Last year Burago and Ivanov succeeded to prove the full result: **Theorem 2** (Burago and Ivanov [8]): *If all geodesics on (T^n, g) $(n \geq 2)$ have no conjugate points than the metric is flat.* The beautiful proof uses tools of integral geometry, in particular, the stable norm, which in this case is shown to be Euclidean. **5. Minimal foliation of codimension 1 on T^d** The attempts to generalize the Aubry–Mather theory met with essential difficulties. Actually, Mather succeeded to give such a generalization which, however, is more complicated and, by necessity, less specific. Another direction was taken in [18], [19] where minimal foliations of codimension 1 on a torus T^d were studied. That means that the leaves of the foliations are hypersurfaces of dimension $d-1 \geq 2$ which satisfy elliptic partial differential equations, the Euler equation of a variational problem. Thus we move outside the realm of dynamical systems, and the considered minimals are viewed as a generalization of the minimal energy states in the models of solid static physics considered by Aubry. Instead of 1-dimensional crystals we consider d-dimensional ones. We restrict ourself to the non-parametric situation and describe the minimals as a graph

$$x_d = u(x) \; ; \quad x = (x_1, \ldots, x_n) \; ; \quad n = d-1$$

where $x_1, x_2, \ldots, x_d$ are the coordinates on $\mathbb{R}^d$ and $T^d = \mathbb{R}^d/\mathbb{Z}^d$. The variational problem has the form

$$\int F(x, u, u_x)\, dx \; ; \quad dx = dx_1 \wedge \cdots \wedge dx_n$$

where the integrand $F(x, x_d, p)$ is periodic in the first d coordinates, strictly convex and of quadratic growth in p. A typical and relevant example is

$$F = \frac{1}{2}|p|^2 + V(x,u) \; ; \quad V \in C^\infty(T^d)$$

where V represents a periodic potential. One can define minimal foliations similarly as above and is led to *minimals without selfintersections*: A function u is called a minimal if

$$\int_{\mathbb{R}^n} \left(F(x, u+\varphi, u_x + \varphi_x) - F(x, u, u_x) \right) dx \geq 0$$

for all $\varphi \in C^1_{\mathrm{comp}}(\mathbb{R}^n)$. It is said to be without selfintersection if

$$u(x+j) - u(x) - j_d \neq 0 \quad \text{for all} \quad (j, j_d) \in \mathbb{Z}^d \backslash (0) \; .$$

We remark, that for $d = 2$ every minimal is automatically without selfintersections, but for $d > 2$ this is a necessary and independent requirement. In this framework an analogue of the theory of Aubry–Mather can be developed. The key to this theory is an inequality valid for minimals without selfintersections, quite similar to the trapping property on T^2. **Theorem 3** ([18], [19]).

There exists a constant D, depending on F only such that any minimal without selfintersections lies in a slab

$$|x_d - (\alpha \cdot x) - \mathrm{const}| \leq \frac{1}{2} D \sqrt{1 + \alpha^2}$$

of width D, where $\alpha \in \mathbb{R}^n$. Moreover, any such slabs contains a minimal without selfintersection. This is a pointwise a priori estimate leading to the compactness properties giving the required existence theorems. The proof makes use of techniques as they were developed by di Giorgi in his classical work (1957) on regularity theory. That these methods can be used for minimals, and even for quasi-minimals, was noted by Giaquinta and Giusti [11]. **6. Hyperbolic geometry and quasi-minimals** These ideas play a prominent role in hyperbolic geometry. The first case of the trapping property goes back to M. Morse [17] in 1924 who used the term "geodesics of class A" for minimals. We describe right away the extension of Morse's result to n-dimensional hyperbolic space $(\mathbb{H}^n, g_0)$ where g_0 is the standard non-Euclidean metric in the unit ball B^n. We are interested in k-dimensional submanifolds M^k in $\mathbb{H}^n$ which are minimal with respect to the k-dimensional volume form for a metric g comparable to g_0 in the sense that

$$(*) \qquad c^{-1} g_0 \leq g \leq c\, g_0$$

holds for some constant $c \geq 1$. Thus the curvature of g may have varying sign. To formulate the result we need the concept of a λ-quasi-minimal: A manifold M^k is λ-quasi-minimal if for any compact k-dimensional subset $\Omega \subset M^k$ one has,

$$\mathrm{vol}_k \Omega \leq \lambda \mathrm{vol}_k \Omega'$$

for any $\Omega' \subset \mathbb{H}^n$ with $\partial\Omega' = \partial\Omega$ and in the same homotopy class as Ω. For $\lambda = 1$ we obtain the concept of a minimal. It is easy to see that any minimal for the metric g is a λ-quasi-minimal for the standard metric g_0, provided $\lambda = c^2 \geq 1$. This allows us to eliminate the metric g by studying quasi-minimals in $(\mathbb{H}^n, g_0)$. The relevant trapping property for quasi-minimals in $(\mathbb{H}^n, g_0)$ is due to V. Bangert and U. Lang, who proved last year the **Theorem 4** (Bangert–Lang [6]): *If k-dimensional Ω is a λ-quasi-minimals for g_0 then there exists a constant $D = D(\lambda, k)$ such that*

$$\mathrm{dist}_{g_0}(p, C) \leq D \quad \textit{for all} \quad p \in \Omega$$

where C is the convex hull of $\partial\Omega$. For $k = n-1$ the best constant is given by (see [14])

$$D = \frac{\pi}{2}\sqrt{\lambda^2 - 1}\,.$$

The proof uses geometric measure theory and generalizations of the isoperimetric inequality. For $n = 2$ this theorem implies that any λ-quasi-minimal remains in a non-Euclidean neighborhood fixed by D of a non-Euclidean line (i.e. a circle). Since this neighborhood pinches on the endpoints on ∂B^n we conclude that λ-quasi-minimal have limits at infinity. This is essentially the contents of Morse's classical result. This theorem can, and has, been used to solve the "asymptotic Plateau problem" where one looks for a minimal with respect to a metric g with given boundary values at infinity. This generalizes the theorem by Anderson [1] from the hyperbolic metric g_0 to any comparable one, in the sense of $(*)$. This permits, in particular, sectional curvature of varying signs. It is not our aim to describe this interesting theory. We wanted to show how the idea of minimals and the corresponding trapping property enter in these different areas. It seems that the last mentioned situation of hyperbolic geometry is most natural for this principle.

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