Quantum Stochastic Calculus

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Detereministic classical dynamics, reversible quantum dynamics in the Schrödinger - Heinsenberg pictures and irreversible dynamics of diffusion processes driven by Brownian motion admit 'infinitesimal' or 'differential' descriptions which can be expressed in terms of derivations of * algebras. One of the central aims of quantum stochastic calculus is to unify all these and explore for similar algebraic features in the differential description of irreversible quantum dynamical systems as well as classical Markov chains.

Observables in quantum theory are, usually, selfadjoint operators in a Hilbert space. A *process* of observables is described by a map $t \rightarrow X(t)$ where t denotes time and $X(t)$ is a selfadjoint operator. Our aim is to describe such processess differentially as $dX(t) = \sum_i L_i(t) d\Lambda_i(t)$ where $d\Lambda_i$ are some 'universal' differentials including, of course, dt. In the case of irreversible diffusions $d\Lambda_i$'s different from dt are differentials of independent Brownian motion processes. In the context of quantum theory it is natural to explore the possibility of using the creation, annihilation and number operator processes of free field theory as candidates for the universal differentials. The free field operators in the boson Fock space $\Gamma(h \otimes L^2(I\!\!R^+))$ can be expressed in terms of a chosen orthonormal basis $\{e_i\}, i = i = 1, 2, ...,$ in h as $\{\Lambda^\alpha_\beta(t), t \geq 0,\ \alpha, \beta \in \{0, 1, 2, ...,\}\}$ obeying the (extended) canonical commutation relations (ECCR) :

$$\Lambda^0_0(t) = tI; \quad \Lambda^\alpha_\beta(t)^\dagger = \Lambda^\beta_\alpha(t);$$

$$[\Lambda^\alpha_\beta(s),\ \Lambda^\mu_\nu(t)] = \hat{\delta}^\alpha_\nu\ \Lambda^\mu_\beta(s\wedge t)$$

where $\hat{\delta}^\alpha_\nu = 0$ if $\alpha = 0$ or $\beta = 0$ or $\alpha \neq \beta$ and $= 1$ otherwise. $\Lambda^i_0(t) = a(e_i \otimes 1_{[0,t]})$, $\Lambda^0_t(t) = a^\dagger(e_i \otimes 1_{[0,t]})$ are the annihilation and creation operators whereas $\Lambda^i_j(t), 1 \leq i,j < \infty$ are derived from the infinitesimal versions of second quantization operators. Furthermore all the operators $\Lambda^\alpha_\beta(t) - \Lambda^\alpha_\beta(s)$ satisfy the locality property that they live in the factor $\Gamma(h \otimes L^2[s,t])$ in the tensor product decomposition :

$$\mathcal{H} = \Gamma(h \otimes L^2(I\!R_+)) = \Gamma(h \otimes L^2[0,s]) \otimes \Gamma(h \otimes L^2[s,t]) \otimes \Gamma(h \otimes L^2[t,\infty)).$$

(We recall that $\Gamma(k)$ denotes the boson Fock space $\mathbb{C} \oplus k \oplus k^{s^2} \oplus \cdots \otimes k^{s^n} \oplus \cdots s^n$ denoting n-fold symmetric tensor product). ECCR and locality facilitate the development of 'quantum stochastic integrals' of the form $\int\limits_0^t L^\alpha_\beta(s) \otimes d\Lambda^\beta_\alpha(s)$ in Hilbert spaces of the type $\hat{\mathcal{H}} = h_0 \otimes \Gamma(h \otimes L^2(I\!R_+))$ where $L^\alpha_\beta(s)$ is an operator in the factor $h_0 \otimes \Gamma(h \otimes L^2[0,s])$ and repeated index implies summation over the index. Consider two processes $X(t), Y(t)$ of operators in $\hat{\mathcal{H}}$ satisfying

$$\begin{aligned} X(t) &= X(0) + \int\limits_0^t L^\alpha_\beta(s) \otimes d\Lambda^\beta_\alpha(s), \\ Y(t) &= Y(0) + \int\limits_0^t M^\alpha_\beta(s) \otimes d\Lambda^\beta_\alpha(s), \end{aligned}$$

expressed in differential notation as

$$dX = L^\alpha_\beta d\Lambda^\beta_\alpha,\ dY = M^\alpha_\beta d\Lambda^\beta_\alpha.$$

Then one has the 'quantum Ito's formula' [H-P1]

$$\left.\begin{aligned} d(XY) &= (dX)Y + X(dY) + dX.dY, \\ dX.dY &= L^\alpha_\beta M^\mu_\nu d\Lambda^\beta_\alpha d\Lambda^\nu_\mu, \\ d\Lambda^\beta_\alpha \Lambda^\nu_\mu &= \hat{\delta}^\beta_\mu d\Lambda^\nu_\alpha. \end{aligned}\right\} \tag{1}$$

This implies, in particular, that

$$(d\Lambda_0^i + d\Lambda_i^0)^2 = dt,\ i = 1, 2, ... \tag{2}$$

If

$$d\Lambda_i^i + \sqrt{\lambda}(d\Lambda_0^i + d\Lambda_i^0) + \lambda dt = dN_\lambda^i(t), \lambda > 0, i = 1, 2, ... \tag{3}$$

then

$$(dN_\lambda^i)^2 = dN_\lambda^i. \tag{4}$$

Remarkably, (2) turns out to be the classical Ito's formula for Brownian motion and (4) the same for a Poisson process of intensity λ. Thus *the classical probabilistic correction formulae for products of stochastic differentials are consequences of the canonical commutation relations of the free Bose-Einstein field.*

If dim $h = 1,\ \Lambda_0^1 = A, \Lambda_1^0 = A^\dagger$ and $\Lambda_1^1 = \Lambda$, write

$$F(t) = \int_0^t (-1)^{\Lambda(s)} dA(s),\ F^\dagger(t) = \int_0^t (-1)^{\Lambda(s)} dA^\dagger(s).$$

Then $F, F^\dagger$ obey the canonical anticommutation relations (CAR) :

$$F(t)F(s) + F(s)F(t) = 0,\ \ F(s)F^\dagger(t) + F^\dagger(t)F(s) = s \wedge t.$$

This shows that fermion free field operators can be expressed as quantum stochastic integrals of a reflection-valued process with respect to the boson free field differentials. This is a continuous analogue of the classical Jordan-Wigner transformation [H-P2].

Let $\mathcal{A}$ denote the C^* algebra of observables concerning a quantum system. Assume that $\mathcal{A}$ is realised as a subalgebra of $\mathcal{B}(h_0)$. When the system is coupled to "noise" or "heat bath" any observable $X \in \mathcal{A}$ changes to a new observable $j_t(X)$ after time t which can be viewed as an element of $\mathcal{B}(h_0 \otimes \mathcal{H})$ where the map $X \to j_t(X)$ is a C^* endomorphism. Following the classical theory of diffusion processes it is possible to formulate a "quantum stochastic differential equation" for $\{j_t\}$:

$$dj_t(X) = j_t(\theta^\alpha_\beta(X))d\Lambda^\beta_\alpha(t),\ j_0(X) = X \otimes I,\ X \in \mathcal{A}, \tag{5}$$

where $\theta^\alpha_\beta : \mathcal{A} \to \mathcal{A}$ are linear maps which may be unbounded. The hypothesis that j_t is an endomorphism together with the quantum Ito's formula implies that

$$\theta^\alpha_\beta(XY) - \theta^\alpha_\beta(X)Y - X\theta^\alpha_\beta(Y) = \theta^\alpha_j(X)\theta^j_\beta(Y) \tag{6}$$

where $j = 1, 2, ...$ and $\alpha, \beta \in \{0, 1, 2, ...\}$. The right hand side measures the "deviation" from the property of θ^α_β being a derivation. If we write

$$\begin{aligned} \pi(X) &= (\theta^i_0(X))_{i=1,2,....} \text{ as a column,} \\ \rho(X) &= ((\theta^i_j(X) + \delta^i_j X))_{i,j\in\{1,2,...\}} \end{aligned}$$

as a matrix then ρ is a representation of $\mathcal{A}$ and

$$\pi(XY) = \pi(X)Y + \rho(X)\pi(Y),$$

i.e., π is a "twisted" ρ-derivation of $\mathcal{A}$. M.P. Evans [E] established the existence of a unique solution $\{j_t\}$ of endomorphisms of $\mathcal{A}$ satisfying (5) when the maps θ^α_β are bounded and dim $h < \infty$. The case dim $h = \infty$ was settled by A. Mohari and K.B. Sinha [M-S].

Connections between (6) and the Hochschild cohomology were investigated by R.L.Hudson [H].

When $\mathcal{A}$ is the abelian C^* algebra $\ell^\infty(I\!N)$ and $\theta^i_j(f) = \delta^i_j(f(i) - f)$, $\theta^i_0(f) = (f(i) - f)h_i, \theta^0_i(f) = (f(i) - f)\bar{h}_i, \theta^0_0(f) = \sum\limits_i |h_i|^2(f(i) - f)$ where $\sum\limits_i |h_i(j)|^2 < \infty, h_i \in \ell^\infty(I\!N)$ it follows as a corollary of the Mohari-Sinha extension of Evan's theorem that $\{j_t\}$ is well-defined as a solution of (5) and it describes the minimal Markov chain with countable state space $I\!N$ and infinitesimal generator θ^0_0. Thus Markov chains can be described in terms of quantum stocahstic differential equations with respect to the universal differentials $d\Lambda_0^i, d\Lambda_i^0$ and $d\Lambda_i^i, i = 1, 2, ...$ and dt.

For a self-contained introduction to the matter under discussion we refer to the book [P]. A lively discussion of quantum probability, stochastic calculus and related problems may be found in [M].

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