Lecture about Pierre-Louis Lions [1]

Pierre-Louis Lions *wurde für seine glänzenden Beiträge zur Theorie der nichtlinearen partiellen Differentialgleichungen mit der Fields-Medaille ausgezeichnet. Eines der Hauptziele in diesem Gebiet ist Entwicklung von Theorien, welche Existenz, Eindeutigkeit und Stabilität von Lösungen für möglichst umfassende Klassen von Gleichungen sicherstellen. Pierre-Louis Lions spielte die führende Rolle bei der Entstehung einer solchen Theorie, die heute als „Viskositätsmethode“ bekannt ist. Sie zeichnet sich durch besondere Eleganz und umfassende Einsetzbarkeit aus und erlaubt darüberhinaus die Behandlung zahlreicher Beispiele, die in den Anwendungen von großer Bedeutung sind. Zusätzlich ist ihm in den letzten Jahren ein wichtiger Durchbruch in der Theorie der Boltzmann-Gleichung und ähnlicher Transportgleichungen gelungen. Ohne Zweifel hat Lions die führende Rolle bei der Entwicklung des Gebiets in den letzten fünfzehn Jahren eingenommen.*
Auszug aus der Laudatio von **Srinivasa S.R. Varadhan**.

Pierre Louis Lions has made unique constributions over the last fifteen years to mathematics. His contributions cover a variety of areas, from probability theorey to partial differential equations (PDEs). With in PDE he has done several beautiful things in nonlinear equations. The choice of his problems has always been motivated by applications. Many of the problems in Physics, Engineering and Economics when formulated in mathematical terms lead to nonlinear PDEs. They are often very hard problems. The nonlinearity makes each equation different. The work of Lions is important because he has developed techniques that, with variations, can be applied to classes of such problems. Saying something is nonlinear is not saying much. In fact it could even be linear. The entire class of nonlinear PDEs is therefore very extensive and one does not expect an all inclusive theorey. On the other hand one does not want to treat each example differently and have a collection of unrelated techniques. It is thus extremely important to identify large classes that admit a unified treatment.

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In dealing with nonlinear PDEs one has to allow for nonclassical or nonsmooth solutions. Unlike the linear case one cannot use the theory of distributions to define the notion of a weak solution. One has to invent the appropriate notion of a generalized solution and hopefully this would cover a wide class and be sufficient to yield a complete theory of existence, uniqueness and stability for the class.

Due to the very limited time that is available, I shall focus on three areas within nonlinear PDE where Lions has made major constributions. The first is the so called **viscosity method**. This is a long story where, over many years, in partial collaboration with others (mainly Evans, Crandall and Ishii) Lions has developed the method which is applicable to the large class of nonlinear PDEs known as fully nonlinear second order degenerate elliptic PDEs. The class contains very many important subclasses that arise in different contexts.

By a nonlinear PDE one is trying to solve an equation involving an unknown function and its derivatives. Let u be a function in a region G and let $Du, D^2u, ..., D^ku$ be its derivatives of order up to k. A non linear PDE is an equation of the form

$$F[x, u(x), (Du)(x), (D^2u)(x), ..., (D^k(u))(x)] = 0$$

in G with some boundary conditions on ∂G. Such a PDE is said to be nonlinear and of order k. The viscosity method applies in cases where $k = 2$ and $F(x, u, p, H)$ has certain monotonicity properties in the arguments u and H.

More precisely it is nondecreasing in u and nonincreasing in H. Here u is a scalar and H is a symmetric matrix of size $n \times n$ with the natural partial ordering for symmetric matrices.

There are many examples of such functions:

- Linear elliptic equations:

  $$-\sum_{i,j} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + f(x) = 0$$

  where the matrix $a^{ij}(x)$ is uniformly positive definite. In this case the function F is given by

  $$F(x, u, p, H) = -\text{trace}(a(x)H) + f(x)$$

- First order equations:

$$f(x, u(x), (Du)(x)) = 0$$

  These include Hamilton-Jacobi equations where it all started. One added a term of the form $\epsilon\Delta$ to the equation and constructed the solution in the limit as ϵ went to zero. The theory owes its name to its early origins.

If one has a family F_α of such functions one can generate a new one by defining

$$F = \sup_\alpha F_\alpha$$

If one has a two parameter family $F_{\alpha\beta}$ of such functions one can generate a new one by defining

$$F = \sup_\alpha \inf_\beta F_{\alpha\beta}$$

Such examples arise naturally in control theory and game theory and are referred to as Hamilton-Jacobi-Bellman and Isaacs equations.

In order to understand the notion of a generalized solution it is convenient to talk about supersolutions and subsolutions. Suppose u is a subsolution i.e.

$$F(x, u(x), (Du)(x), (D^2u)(x)) \leq 0$$

and we have another function ϕ, which is smooth, such that $u - \phi$ has a maximum at some point $\hat{x}$ then by calculus $Du(\hat{x}) = D\phi(\hat{x})$ and $D^2(u)(\hat{x}) \leq D^2(\phi)(\hat{x})$. From the monotonicity properties of F it follows that

$$F(\hat{x}, u(\hat{x}), (Du)(\hat{x}), (D^2u)(\hat{x})) \geq \\ F(\hat{x}, u(\hat{x}), (D\phi)(\hat{x}), (D^2\phi)(\hat{x})).$$

Therefore

$$F(\hat{x}, u(\hat{x}), (D\phi)(\hat{x}), (D^2\phi)(\hat{x})) \leq 0.$$

The last inequality makes sense without smoothness assumption on u. We can try to define a nonsmooth subsolution as a u that satisfies the above for arbitrary smooth ϕ and $\hat{x}$ provided $u - \phi$ has a maximum at $\hat{x}$. The definition of a super solution is similar and a solution is one that is simultaneously a super and sub solution. Let us consider [first] a Dirichlet boundary value problem where we want to find a u that solves our equation and has boundary value zero. A key step is to establish the comparison theorem that if u is a subsolution and if v is a supersolution in a bounded domain G and if $u \leq v$ on the boundary ∂G then $u \leq v$ in $G \cup \partial G$. From this point on, the theory proceeds in way similar to the classical Perron's method for solving the Dirichlet Problem.

The **second** body of work that I want to discuss has to do with the Boltzmann equation and similar equations. During the last six or seven years Pierre-Louis Lions has played the central role in new developments in the theory of the Boltzmann Equation and similar transport equations. These are important in kinetic theory and arise in a wide variety of physical applications. We will for simplicity stay within the context of the Boltzmann equation. In R^3 we have a collection of particles moving along and interacting through "collisions" among themselves. Since we do not want to keep track of the position and velocities of the particles individually we abstract the situation by the density $f(x, v)$ of particles that are at x with velocity v. Even if there is no interaction, the density $f(x, v)$ will change in time due to uniform motion of the particles. The time dependent density $f(t, x, v)$ will satisfy the equation

$$\frac{\partial f}{\partial t} + v.\nabla_x f = 0.$$

The collisions will change this equation to

$$\frac{\partial f}{\partial t} + v.\nabla_x f = Q(f, f).$$

Here Q is a quadratic quantity that represents binary collisions and its precise form depends on the nature of the interaction. Generally it looks like

$$Q(f, f) = \int \int_{R^3 \times S^2} dv_* d\omega \, B(v - v_*, \omega)\{f' f'_* - f f_*\}$$

The notation here is standard: v and v_* are the incoming velocities and v' and v'_* are the outgoing velocities. B is the collision kernel. For given incoming velocities v and v_*, ω on the sphere S^2 parametrizes all the outgoing velocities compatible with the conservation of energy and momenta.

$$v' = v - (v - v_*, \omega)\omega, \quad v'_* = v + (v - v_*, \omega)\omega$$

and f', f_*, f'_* are $f(t, x, v)$ with v replaced by the correspondingly changed v', v_* and v'_*.

This problem of course has a long history. Smooth and unique solutions had been obtained for small time or globally for initial data close to equilibrium. Carleman had studied spatially homogeneous solutions. But a general global existence theorem had never been proved. The work of Lions (in collaboration with DiPerna) is a breakthrough for this and many other related transport problems of great physical interest.

The **third and final** topic that I would like to touch on is the contribution Lions has made to a class of variational problems. There are many nonlinear PDEs that are Euler equations for variational problems. The first step in solving such equations by the variational method it to show that the extremum is attained. This requires some coercivity or compactness. If the quantity to be minimized has an "energy" like term involving derivatives, then one has control on local regularity along a minimizing sequence. This usually works if the domain is compact. If the domain is noncompact the situation is far from clear. Take for instance the problem of minimizing

$$\int_{R^N} |(\nabla f)(x)|^2 dx - \int\int V(x-y) f^2(x) f^2(y)\, dx dy$$

over functions f with L_2 norm λ (fixed positive number). Here V is a reasonable function decaying at ∞. Because of translation invariance, the minimizing sequence must be centered properly in order to have a chance of converging. The key idea is that, in some complicated but precise sense, if the minimizing sequence cannot be centered, then any member of the sequence can be thought of as two functions with supports very far away from each other. If we denote the infimum by $\sigma(\lambda)$ then along such sequences the infimum will be $\sigma(\lambda_1) + \sigma(\lambda_2)$ with $\lambda_1 + \lambda_2 = \lambda$, $0 < \lambda_1, \lambda_2 < \lambda$ rather than $\sigma(\lambda)$. If independently one can show that $\sigma(\lambda)$ is strictly subadditive then one can prove the existence of a minimizer. This idea has been developed fully and applied successfully by Lions to many important and interesting problems.